

Nonparametric estimation for dependent data with an application to panel time series ¹

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Abstract

In this paper we consider nonparametric estimation for dependent data, where the observations do not necessarily come from a linear process. We study density estimation and also discuss associated problems in nonparametric regression using the 2-mixing dependence measure. We compare the results under 2-mixing with those derived under the assumption that the process is linear.

In the context of panel time series where one observes data from several individuals, it is often too strong to assume the joint linearity of processes. Instead the methods developed in this paper enable us to quantify the dependence through 2-mixing which allows for nonlinearity. We propose an estimator of the panel mean function and obtain its rate of convergence. We show that under certain conditions the rate of convergence can be improved by allowing the number of individuals in the panel to increase with time.

Keywords: Density estimation, nonparametric regression, 2-mixing, nonlinear processes, panel time series.

AMS 2000 subject classifications: Primary: 62G05, 62M10; Secondary: 62G07, 62G08.

1 Introduction

Nonparametric estimation for dependent observations has a long history in statistics. Rosenblatt [1970] first studied density estimation for dependent data. Since then several authors have considered nonparametric estimation under various assumptions. For example, Hall and Hart [1990a], Giraitis et al. [1996], Mielniczuk [1997] and Estevas and Vieu [2003] consider density estimation for linear processes which have long memory, whereas Cheng

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and Robinson [1991] consider density estimation for random variables which are nonlinear functions of a linear process. A notable result, is that they show if the observations were from a linear process and have short memory, then the usual rate of convergence, known for independent observations, also holds for dependent observations. On the other hand, for long memory processes, the rate of convergence is different. Interestingly, despite long memory influencing the rate of convergence, there is no influence of long memory on the bandwidth choice, which is same regardless of short or long memory. In other words, if the observations come from a linear process, a larger bandwidth does not improve the rate of convergence of the density estimator. Similar results can also be derived for nonparametric regression problems (c.f. Hall and Hart [1990b], Cheng and Robinson [1994] and Csörgö and Mielniczuk [1995, 1999, 2001]). However, usually it is assumed that the observations come from a linear process or are functions of a linear process. In the case of linearity, the joint density of the observations can be characterised (in some sense) in terms of the autocovariances. It is this representation that allows for the mean squared error of the nonparametric estimator to be derived in terms of the autocovariance function. However this result does not necessarily hold when the process is nonlinear.

The assumption of linearity can be relaxed by using the notion of 2-mixing (see Bosq [1998]), and in this paper we obtain rates of convergence for processes which are 2-mixing. Unlike the autocovariance function, 2-mixing can be considered as a measure of dependence between two random variables (see Definition 3.1, below) and the *2-mixing size* quantifies this dependence: a large mixing size indicates little dependence, whereas a small mixing size indicates large dependence. The 2-mixing size can be established for several types of processes, for example, linear processes, see Athreya and Pantula [1986], Cline and Pu [1999], Chanda [1974] and the Appendix A.4 (noting that strong mixing implies 2-mixing, though the converse is not necessarily true) and nonlinear processes, see Masry and Tjøstheim [1995], Bousamma [1998] and Basrak et al. [2002]. Assuming that the 2-mixing size is sufficiently large, Bosq [1998] obtains the rate of convergence of several nonparametric estimators. However despite, there being extensive literature on nonparametric estimation for linear processes and some on nonparametric estimation for processes which are 2-mixing with a sufficiently large 2-mixing size, as far as we are aware very little exists on nonparametric estimation for nonlinear processes whose 2-mixing size is not sufficiently large (for example the ARCH(∞) process, which is a nonlinear process and can have a small mixing size). In this paper we address this issue, and consider nonparametric estimation for dependent data and formulate the results in terms of the 2-mixing size. We study both density estimation and also nonparametric regression problems. A natural application of the methodology proposed in this paper is to panel time series, where one observes several individuals over time and associated with each individual are regressors which are known to influence the individual. We note that even in the case that an individual comes from a linear time series, there is no guarantee that the dependence between individuals is also linear. Therefore we quantify the dependence in terms of the 2-mixing size within and

between individuals over time, and consider nonparametric estimation for panel time series within this framework.

In Section 3 we consider kernel density estimation, in particular we obtain the sampling properties of the Rosenblatt-Parzen kernel estimator and obtain a bound for the mean squared error under the assumption that the time series are stationary and 2-mixing. We show that, like the long memory process, the 2-mixing size can influence the rate of convergence. But unlike the long memory process, a much larger 2-mixing size is required to obtain the usual rate of convergence. Moreover, the optimal bandwidths for the bounds obtained are influenced by the 2-mixing size - the smaller the 2-mixing size the larger the bandwidth. We demonstrate that several problems could arise if one were to falsely suppose that observations were from a linear process, when they do not. For example, if the usual optimal bandwidth for linear processes were used on nonlinear processes, the mean squared error may no longer converge to zero. Thus our results give a warning to practitioners who apply well known results for the linear process, without checking whether the process is linear or not.

In Section 4 we consider nonparametric regression for dependent data. We discuss this with reference to two models. First we suppose the response and explanatory variables (X_t, Z_t) satisfy (i) $X_t = \varphi(Z_t) + h(Z_t)\eta_t$, where $\{\eta_t\}$ and $\{Z_t\}$ are independent of each other, and secondly we assume the conditional expectation satisfies (ii) $\mathbb{E}(X_t|Z_t) = \varphi(Z_t)$. We observe that the latter model includes the former model as a special case. We estimate $\varphi(\cdot)$ using the classical kernel estimator and derive rates of convergence similar to those obtained for the density estimator. But in the case of model (i) the rate of convergence depends on two factors, the 2-mixing size of $\{Z_t\}$ and the rate of decay of the autocovariance function of $\{\eta_t\}$, whereas for model (ii) the rate of convergence is determined by the mixing size of the multivariate random process $\{(X_t, Z_t)\}$.

Panel time series are often used to model the relationship and dynamics between several individuals observed over time, and recently Hjellvik et al. [2004] and Mammen et al. [2005] have used nonparametric methods in this context. Typically it is assumed that the dependence between individuals is linear, however this assumption is often too strong, as there could be nonlinear interactions between the individuals. In Section 5 we consider estimation for nonparametric panel time series, but allow for nonlinear dependence between individuals by quantifying their dependence through their 2-mixing sizes. Let $X_{t,i}$ denote the observation of the i th individual at time t , where $i = 1, \dots, N$ and $t = 1, \dots, T$. We also assume that we observe some explanatory variables $Z_{t,i}$ which influence $X_{t,i}$. We suppose the influence is common over all individuals, that is the response and explanatory variables $(X_{t,i}, Z_{t,i})$ satisfy the relation $\mathbb{E}[X_{t,i}|Z_{t,i} = z] = \varphi(z)$ for all $i \in \mathbb{N}$ and $t \in \mathbb{Z}$. We propose a kernel based estimator for $\varphi(\cdot)$ and derive bounds for the deviation. In panel data it is often observed that there is temporal dependence for each individual and also dependence between individuals. We model this by assuming two different 2-mixing sizes. We show that the rate of convergence of the estimator of φ when the number of individuals N is kept

fixed and $T \rightarrow \infty$, is similar, to the rate of convergence of the nonparametric estimator of model (ii) considered in Section 4. However, we show that the rates can be improved if we allow N to increase with T . Furthermore, if the mixing size is sufficiently large we can obtain the usual nonparametric rate of convergence obtained for iid random variables.

All the proofs can be found in the appendix. Also some 2-mixing inequalities for linear processes used here are included in the appendix.

2 Notation

In this section we introduce some definitions that will be used in the paper. Note we will assume all the necessary densities exist.

We start by defining the multiplicative kernel.

DEFINITION 2.1. *For all $w = (w_1, \dots, w_d) \in \mathbb{R}^d$, K is a multiplicative kernel (see Scott [1992]) of order r , i.e. $K(w) = \prod_{i=1}^d \ell(w_i)$ where ℓ is a univariate, even function such that*

$$\int du \ell(u) = 1, \quad \int du u^i \ell(u) = 0$$

for all $i = 1, \dots, r-1$ and there exists a constant S_K such that

$$[\int du |u|^r \ell(u)]^d = S_K.$$

Let $K_b(z) := b^{-d} K(z/b)$, where $b > 0$ is a bandwidth. Below we define the smoothness class which we use to bound the bias of the estimators.

DEFINITION 2.2. *For $s, \Delta > 0$, the space $\mathfrak{G}_{s,\Delta}^d$ is the class of functions $g : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying: g is everywhere $(m-1)$ -times partially differentiable for $m-1 < s \leq m$; where for some $\rho > 0$ and for all x , the inequality*

$$\sup_{y: |y-x| < \rho} \frac{|g(y) - g(x) - Q(y-x)|}{|y-x|^s} \leq \Delta,$$

holds true with $Q = 0$ when $m = 1$ and for $m > 1$, Q is an $(m-1)$ th-degree homogeneous polynomial in $y-x$, whose coefficients are the partial derivatives of g of orders 1 to $m-1$ evaluated at x ; and Δ is a finite constant.

For brevity, we use the standard notation \wedge to denote minimum and \vee to denote maximum.

3 Kernel density estimation

Suppose we observe the stationary time series $\{Z_1, \dots, Z_T\}$, and let f denote the density of Z_t . The most popular estimator of f , is the Rosenblatt-Parzen kernel estimator

$$\hat{f}(u) = \frac{1}{T} \sum_{t=1}^T K_b(Z_t - u), \tag{3.1}$$

where $K_b(z) = b^{-1}K(\frac{z}{b})$, $b > 0$ is a bandwidth, and K is a multiplicative kernel (see Definition 2.1). In this section we investigate the sampling properties of the kernel density estimator defined above. The dependence of the process $\{Z_t\}$ is quantified in terms of its 2-mixing size.

DEFINITION 3.1. (i) A process $\{Y_t\}$ is said to be 2-mixing with size \mathfrak{v} if for all $t \neq \tau$

$$\sup_{A \in \sigma(Y_t), B \in \sigma(Y_\tau)} |P(A \cap B) - P(A)P(B)| \leq C|t - \tau|^{-\mathfrak{v}}.$$

for some $C < \infty$ independent of t and τ .

(ii) The covariance of a stationary process $\{Y_t\}$ has size \mathfrak{u} if for all $t \neq \tau$, $|\text{cov}(Y_t, Y_\tau)| \leq C|t - \tau|^{-\mathfrak{u}}$ for some $C < \infty$ independent of t and τ .

We note that the covariance is a measure of linear dependence, whereas 2-mixing is a generalisation of this, and can be considered as a measure of dependence. 2-mixing is quite a general notion, which is satisfied by several processes. For example, under certain conditions on the innovations, most linear models are 2-mixing (see Appendix A.4, and Athreya and Pantula [1986], Cline and Pu [1999] and Chanda [1974], where strong mixing is shown). Further, under additional conditions on the innovations and the parameters, ARCH/GARCH processes are also strongly mixing (c.f. Masry and Tjøstheim [1995], Bousamma [1998] and Basrak et al. [2002]) which implies that they also 2-mixing. Most of the results and bounds in this paper are derived using 2-mixing. In general, the larger the mixing size the faster the rate of convergence. For example, in the case of iid observations (the 2-mixing size can be treated as ∞) using just a few observations, information over the entire domain of the density function can be obtained. On the other hand, a sample which has a small mixing size (so tends to be clustered about certain points) will require a much larger number of observations to give the same information.

We first derive a bound for the mean squared error (MSE) $\mathbb{E}|\hat{f}(z) - f(z)|^2$ using only minimal assumptions on the distribution of $\{Z_t\}$.

PROPOSITION 3.1. Suppose the stationary process $\{Z_t\}$ is 2-mixing with size \mathfrak{v} and the marginal density f of Z_t and its second derivative f'' are both uniformly bounded. Let \hat{f} be defined as in (3.1), where K is a rectangular kernel, i.e., $K(x) = 1$ if $x \in [-1/2, 1/2]$ and zero otherwise. Then we have

$$\mathbb{E}|\hat{f}(z) - f(z)|^2 = O(b^4 + T^{-[\mathfrak{v} \wedge 1]} b^{\frac{-(\mathfrak{v} \vee 1) + 1}{\mathfrak{v} \vee 1}}) = \begin{cases} O(b^4 + T^{-1} b^{-\frac{\mathfrak{v} + 1}{\mathfrak{v}}}) & \mathfrak{v} > 1 \\ O(b^4 + T^{-\mathfrak{v}} b^{-2}) & \mathfrak{v} \leq 1 \end{cases}$$

PROOF. To prove the result we will bound the risk using the standard variance bias decomposition. First the bias: as we are using a rectangular kernel and f'' is uniformly bounded, it is clear that $\mathbb{E}(\hat{f}(z)) = f(z) + O(b^2)$. To obtain a bound for the variance we require a bound for the covariances inside the variance expansion $T^2 \cdot \text{var}(\hat{f}(z)) = \sum_{t, \tau} \text{cov}[K_b(Z_t - z), K_b(Z_\tau - z)]$. Since $\{Z_t\}$ is 2-mixing with size \mathfrak{v} by using the covariance

inequality in Bradley [1996] (see also Rio [1993]) we have

$$\begin{aligned} & |\text{cov}[K_b(Z_t - z), K_b(Z_\tau - z)]| \\ & \leq 4 \cdot \int_0^\infty \int_0^\infty \min(C|t - \tau|^{-\mathfrak{v}}, P(|K_b(Z_t - z)| > x), P(|K_b(Z_\tau - z)| > y)) \, dx dy. \end{aligned} \quad (3.2)$$

Studying $P(|K_b(Z_t - z)| > x)$ and recalling that $K(\cdot)$ is a rectangular kernel we can show that

$$P(|K_b(Z_t - z)| > x) = \begin{cases} 0, & \text{if } x > 1/b; \\ P(Z_t \in [z - b/2, z + b/2]), & \text{otherwise.} \end{cases}$$

By using the mean value theorem we have $P(X_t \in [z - b/2, z + b/2]) = bf(\tilde{z})$, for some $\tilde{z} \in [z - b/2, z + b/2]$. Substituting this into (3.2) leads to

$$\begin{aligned} |\text{cov}[K_b(Z_t - z), K_b(Z_\tau - z)]| & \leq 4 \cdot \int_0^{1/b} \int_0^{1/b} \min(C|t - \tau|^{-\mathfrak{v}}, b \cdot f(\tilde{z})) \, dx dy \\ & = 4 \cdot b^{-2} \min(C|t - \tau|^{-\mathfrak{v}}, b \cdot f(\tilde{z})). \end{aligned} \quad (3.3)$$

Altogether this yields the bound

$$T^2 \cdot \text{var}(\hat{f}(z)) \leq 4 \sum_{t, \tau} b^{-2} \min(C|t - \tau|^{-\mathfrak{v}}, b \cdot f(\tilde{z})).$$

Examining the minimum inside the summand above, we partition the sum into two parts which we bound separately (for the details see the proof of Theorem 3.2, in the Appendix). Finally recalling that $[\mathbb{E}(\hat{f}(z)) - f(z)]^2 = O(b^4)$ leads to the desired result. \square

We observe, in the proof above, that besides the 2-mixing condition we do not have any assumptions on the joint distribution of (Z_t, Z_τ) . The cost of using such weak assumptions is that the usual bound $O(b^4 + (bT)^{-1})$ for the MSE, obtained for independent observations, is not achieved. Even for large \mathfrak{v} the 2-mixing size has an influence on the bound. However, introducing some assumptions on the joint densities of $\{Z_t\}$ allows us to tighten the bound derived in (3.3) and, hence for a sufficiently large mixing size \mathfrak{v} to recover the usual bound $O(b^4 + (bT)^{-1})$ for the MSE (we note that the rest of the proofs in this section and the subsequent sections require more subtle arguments, and these can be found in the appendix).

ASSUMPTION 3.1 (Densities and kernels). (i) *The marginal density f is uniformly bounded.*

(ii) *For each $t, \tau \in \mathbb{Z}$ let $f^{(t, \tau)}$ denote the joint density of (Z_t, Z_τ) . Define¹ $F^{(t, \tau)} := f^{(t, \tau)} - f \otimes f$. Then $\|F^{(0, t)}\|_{p_F}$ is uniformly bounded in t for some $p_F > 2$ and we define $q_F = 1 - 2/p_F$.*

(iii) *The univariate kernel K is uniformly bounded and has a finite first and second moment, i.e., $\|K\|_1 < \infty$ and $\|K\|_2 < \infty$.*

¹We use the notation $f \otimes g(x, y) = f(x)g(y)$ and $\|f\|_p = (\int |f(x)|^p dx)^{1/p}$.

We use these assumptions to derive a bound for the MSE of the density estimator.

THEOREM 3.2. *Let us suppose the stationary time series $\{Z_t\}$ is 2-mixing with size \mathfrak{v} and Assumption 3.1 is fulfilled for some $q_F \in (0, 1)$.*

Let \hat{f} be defined as in (3.1), where K is a univariate kernel of order r . In addition assume that $f \in \mathfrak{G}_{s,\Delta}^1$ for some $\Delta, s > 0$ (see Definition 2.2), and let $\rho = r \wedge s$. Then we have for all $z \in \mathbb{R}$

$$\mathbb{E}|\hat{f}(z) - f(z)|^2 = O\left(b^{2\rho} + b^{-1} \cdot T^{-1} + b^{-2-q_F(1-\lfloor \mathfrak{v} \vee 1 \rfloor)} \cdot T^{-\lfloor \mathfrak{v} \wedge 1 \rfloor}\right), \quad T \rightarrow \infty.$$

For ease of presentation we have only stated the result for univariate $\{Z_t\}$, however it is straightforward to extend this result for multivariate $\{Z_t\}$. Indeed, the proof of the theorem given in the Appendix is derived for random vector $\{Z_t\}$ (as we require the multivariate case in Section 4).

REMARK 3.1. We note that in the bound given in Theorem 3.2 the second term dominates the third term when $\mathfrak{v} > 1 + 1/q_F$. Conversely, when $\mathfrak{v} < 1 + 1/q_F$ the third term dominates the second term. Moreover, the third term can be partitioned into two further cases, when $1 < \mathfrak{v} \leq 1 + 1/q_F$ and when $\mathfrak{v} \leq 1$. This means that Theorem 3.2 can be written as

- (i) if $\mathfrak{v} > 1 + 1/q_F$, then $\mathbb{E}|\hat{f}(z) - f(z)|^2 = O\left(b^{2\rho} + \frac{1}{bT}\right)$;
- (ii) if $1 < \mathfrak{v} \leq 1 + 1/q_F$, then $\mathbb{E}|\hat{f}(z) - f(z)|^2 = O\left(b^{2\rho} + \frac{1}{b^{(2+q_F(1-\mathfrak{v}))}T}\right)$;
- (iii) if $\mathfrak{v} \leq 1$ then $\mathbb{E}|\hat{f}(z) - f(z)|^2 = O\left(b^{2\rho} + \frac{1}{b^2T^\mathfrak{v}}\right)$;

as $T \rightarrow \infty$.

Studying the three bounds, we see that the bound increases linearly with \mathfrak{v} for $0 \leq \mathfrak{v} \leq 1$, after this point there is a change in behaviour and the increase is more gradual. The bound plateaux when $\mathfrak{v} > 1 + 1/q_F$, after this point we have the usual nonparametric bound obtained for iid observations. There is also a continuity in the three bounds. More precisely, when \mathfrak{v} is at the boundary of 1 and $1 + q_F^{-1}$, there is a continuous transition between the bounds. \square

We now consider the rate of convergence using the optimal bandwidth b^* .

COROLLARY 3.3. *Suppose the assumptions in Theorem 3.2 are satisfied and $r \geq s$. Let $b^* \approx T^{-\gamma/(2s+1)}$ with*

$$\gamma := \begin{cases} 1, & \mathfrak{v} > 1 + 1/q_F; \\ \lfloor \mathfrak{v} \wedge 1 \rfloor \cdot \frac{2s+1}{2s+(2+q_F(1-\lfloor \mathfrak{v} \vee 1 \rfloor))}, & 1 + 1/q_F \geq \mathfrak{v}. \end{cases} \quad (3.4)$$

Then for all $z \in \mathbb{R}$ we have $\mathbb{E}|\hat{f}(z) - f(z)|^2 = O\left(T^{-\frac{2s}{2s+1} \cdot \gamma}\right)$ as $T \rightarrow \infty$.

In other words, if $b^* \approx T^{-\gamma/(2s+1)}$, then we have

$$\mathbb{E}|\hat{f}(z) - f(z)|^2 := \begin{cases} O\left(T^{-\frac{2s}{2s+1}}\right), & \mathfrak{v} > 1 + 1/q_F; \\ O\left(T^{-\frac{2s}{2s+1} \cdot \left(\frac{2s+1}{2s+(2+q_F(1-\mathfrak{v}))}\right)}\right), & 1 + 1/q_F \geq \mathfrak{v} > 1; \\ O\left(T^{\mathfrak{v} \cdot \frac{2s+1}{2s+2}}\right), & 1 \geq \mathfrak{v}. \end{cases} \quad (3.5)$$

We note that if $\sup_z |f(z)| < \infty$ and $\sup_t \sup_z |f^{(0,t)}(z)| < \infty$ (both the density and the joint densities are uniformly bounded), then uniformly in all t , $\|F^{(0,t)}\|_\infty < \infty$. This means $q_F = 1$, and the bound can be divided into the three cases where $\mathfrak{v} \leq 1$, $1 \leq \mathfrak{v} \leq 2$ and $\mathfrak{v} \geq 2$. On the other hand when $\|F^{(0,t)}\|_{p_F} < \infty$ for only a finite p_F , then $q_F < 1$ and $\mathfrak{v} > 1 + q_F^{-1} > 2$ to be sure of the usual nonparametric bound.

Referring to Corollary 3.3, we observe that when $\mathfrak{v} < 1 + q_F^{-1}$, then the optimal bandwidth b^* is much larger than usual optimal bandwidths encountered in nonparametric regression ($b \approx T^{\frac{-1}{2s+1}}$). We discuss this further in Section 3.2.

3.1 A comparison of the MSEs for linear processes

In this section we compare the MSE in Theorem 3.2 with the results obtained under the stronger condition that the observations $\{Z_t\}$ come from a linear process. We will use the results in Appendix A.4 and show that if the process were linear, and not just mixing, that then the rate of convergence is better than the rate obtained in Corollary 3.3. However, in Section 3.2 we demonstrate that by misspecifying the process to be linear, can lead to several problems with the density estimator, including bounds which do not converge to zero.

Let us suppose $\{Z_t\}$ has a linear process representation and satisfies

$$Z_t = \sum_{j=0}^{\infty} a_j \varepsilon_{t-j}, \quad (3.6)$$

where the innovations $\{\varepsilon_t\}$ are iid. Under the assumptions in Lemma A.11 (see the Appendix), it can be shown that $\text{cov}(K_b(Z_0), K_b(Z_t)) = O(\text{cov}(Z_0, Z_t))$. Using this as the basis, Hall and Hart [1990a], Giraitis et al. [1996], Mielniczuk [1997] and Estevas and Vieu [2003] have shown that if the kernel is of order $r \geq s$ (where r is the order of the kernel and s is the smoothness of f , see Definitions 2.1 and 2.2), the MSE is

$$\mathbb{E}|\hat{f}(z) - f(z)|^2 = O\left(b^{2s} + \frac{1}{bT} + \frac{1}{T}R_T\right), \quad (3.7)$$

where $R_T = \sum_{t=1}^T |\text{cov}(Z_0, Z_t)|$. It is clear that both $\text{cov}(Z_0, Z_t)$ and R_T depend on the rate of decay of the parameters a_j . We observe if $|a_j| \leq Cj^{-\theta}$, then

$$\begin{aligned} |\text{cov}(Z_0, Z_t)| &= O(T^{-2\theta+1}) & \text{and} & & R_T &= O((\log T)T^{-(2\theta-1)+1}) & \text{if} & & 1/2 < \theta \leq 1 \\ |\text{cov}(Z_0, Z_t)| &= O(T^{-\theta}) & \text{and} & & R_T &= O(T^{-\theta+1}) & \text{if} & & \theta > 1. \end{aligned}$$

Substituting these rates into (3.7) we see that the bound of the MSE depends on θ . We recall that a process $\{Z_t\}$ is called a short memory process if $\sum_t |\text{cov}(Z_0, Z_t)| < \infty$, otherwise it is called a long memory process. Now studying (3.7) we see that R_T does not depend on the bandwidth b . In other words long memory has *no influence* on the choice of the optimal bandwidth. To summarise, the rate of convergence for observations coming from a linear process is

$$\mathbb{E}|\hat{f}(z) - f(z)|^2 \leq \begin{cases} O(T^{-(2\theta-1)}), & \text{if } 2\theta - 1 \leq \frac{2s}{2s+1}; \\ O(T^{-\frac{2s}{2s+1}}), & \text{if } 2\theta - 1 > \frac{2s}{2s+1}. \end{cases} \quad (3.8)$$

We now compare these results to (3.5), in particular when $\mathfrak{v} \leq 1$, we have

$$\mathbb{E}|\hat{f}(z) - f(z)|^2 = O(T^{-\mathfrak{v} + \frac{2\mathfrak{v}}{2s+2}}). \quad (3.9)$$

It is difficult to directly compare (3.8) and (3.9), since (3.8) is in terms of its long memory parameter whereas (3.9) is in terms of its mixing size \mathfrak{v} . However in the special case that $\{Z_t\}$ is Gaussian (and thus linear), there is a one-to-one correspondence, for example, if $2\theta - 1 \leq 1$ then the covariance size and mixing size are the same, and $\mathfrak{v} = (2\theta - 1)$. Noting that the Gaussian density is analytic, the rate of convergence is determined by the order of the kernel r . In this case, the rates in (3.8) are better than those in (3.9), but as the order r increases the two rates become close. We illustrate the case when the mixing and the covariance sizes are the same in Figure 1 (for both large and small $r \wedge s$). In the non-Gaussian case, where the 2-mixing and covariance size do not necessarily coincide, $\mathfrak{v} \neq (2\theta - 1)$, we have that $(2\theta - 1)\frac{\ell}{2(\ell+1)} \leq \mathfrak{v} \leq (2\theta - 1)\frac{\ell}{(\ell-2)}$, where the innovations satisfy $\mathbb{E}(|\varepsilon_0|^\ell) < \infty$ (see (A.43) in the appendix). In this case it is not clear which rate (3.8) or (3.9) is better. However substituting the lower bound $\mathfrak{v} \geq (2\theta - 1)\frac{\ell}{2(\ell+1)}$ into Corollary 3.3 yields a rate which is less than (3.8). In summary better rates of convergence can often be obtained if the observations come from a linear process. On the other hand, 2-mixing is a *weaker condition, that is satisfied by a far wider class of processes*. We consider below the MSE for processes which are not linear, and show that misspecifying the model, and assuming linearity, when the process is nonlinear could severely affect the MSE.

3.2 The MSE for nonlinear processes

As far as we are aware, theory is required to bridge the gap for processes which are nonlinear but have a small mixing size. One of the main aims of Theorem 3.2 is to fill in the gap in the theory, and to derive a bound for the MSE when the observations come from nonlinear processes with small 2-mixing size.

The joint densities of processes which are nonlinear do not necessarily satisfy the density decomposition in Lemma A.11. Without this result it cannot be shown that $\text{cov}(K_b(Z_0), K_b(Z_t)) = O(\text{cov}(Z_0, Z_t))$, and the rates in (3.8) do not necessarily hold. Instead, to prove the results, under Assumption 3.1, we use classical mixing inequalities to

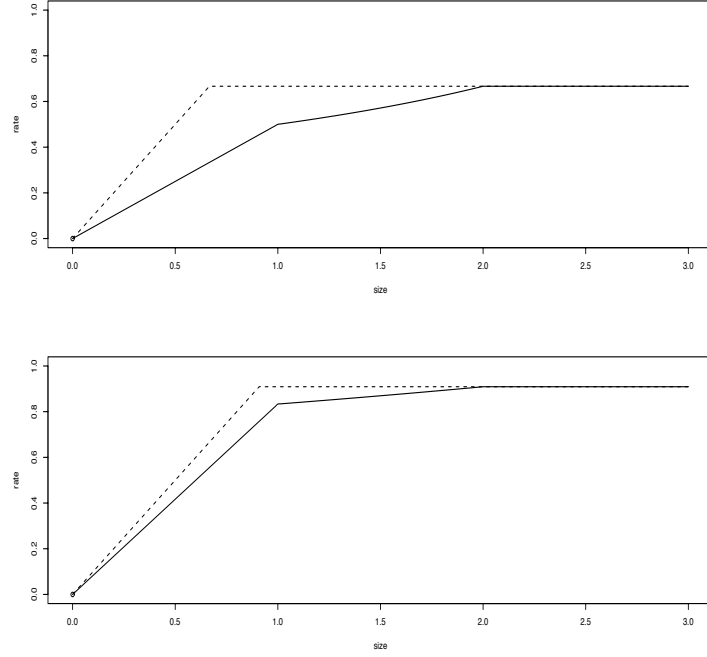


Figure 1: The top and bottom plot corresponds to $\rho = (r \wedge s) = 1$ and $\rho = (r \wedge s) = 5$, respectively. The x -axis is the covariance and mixing size (assuming both are the same) and the y -axis is the indice δ in the MSE $\mathbb{E}|\hat{f}(x) - f(x)|^2 = O(T^{-\delta})$. The solid line is the MSE using 2-mixing and dotted line is the MSE when $\{Z_t\}_t$ is a linear process. We have assumed that $q_F = 1$ (in other words $\|F^{(0,t)}\|_\infty < \infty$).

tighten the bound given in (3.3) (see the proof of Proposition 3.1). More precisely, to prove Theorem 3.2 we show that

$$|\text{cov}(K_b(Z_t - z), K_b(Z_\tau - z))| \leq C \cdot b^{-2} \min(|t - \tau|^{-\mathfrak{v}}, b^{(1+q_F)}),$$

where C is a finite constant (see Lemma A.1, for the proof).

Looking at some of the implications of Theorem 3.2, we demonstrate below that several problems could arise if one were to falsely suppose that the observations come from a linear process, when they do not.

- (i) In the case of linear processes, the optimal bandwidth has the same order as the optimal bandwidth for iid random variables (regardless of long memory). The same is not necessarily true when all that is known is that the process is 2-mixing. Moreover, if the mixing size satisfies $\mathfrak{v} \leq 1$ and the bandwidth is such that $b^2 T^\mathfrak{v} < \infty$, then we see from Theorem 3.2 that the bound does not converge to zero. An important example, is when the ‘usual’ optimal bandwidth for linear or iid data is used (that

is $b \approx T^{-\frac{1}{2s+1}}$). In this case, substituting $b \approx T^{-\frac{1}{2s+1}}$ into Theorem 3.2 leads to the result

$$\mathbb{E}|\hat{f}(z) - f(z)|^2 = \begin{cases} O(T^{-\frac{2s}{2s+1}}) & \mathfrak{v} > 1 + 1/q_F \\ O(T^{\frac{1+q_F(1-\mathfrak{v})-2s}{2s+1}}) & 1 < \mathfrak{v} \leq 1 + 1/q_F \\ O(T^{-\frac{2-\mathfrak{v}(2s+1)}{2s+1}}) & 0 \leq \mathfrak{v} \leq 1 \end{cases}$$

Studying the rates above we see when $\mathfrak{v} < 1 + 1/q_F$, the rates are lower than the rates given using the optimal bandwidth (compare the above with the rates in Corollary 3.3). Moreover, in the case that $\mathfrak{v} \leq \frac{2}{2s+1}$, the bound cannot be used to show consistency of the estimator - since the bound does not even converge to zero.

In short, to estimate the density at any given point, the number of observations (approximately bT) needs to be much larger than in the iid case.

- (ii) Rather surprisingly even when $\sum_{j=1}^{\infty} |\text{cov}(Z_0, Z_j)| < \infty$, the ‘usual $O(T^{-\frac{2s}{2s+1}})$ ’ rate, may not hold, unlike for linear processes. However, the usual rate does hold when $\mathfrak{v} \geq 1 + 1/q_F > 2$. Therefore, even when the mixing and covariance size are the same, a far larger mixing size may be required to obtain the ‘usual $O(T^{-\frac{2s}{2s+1}})$ ’ rate of convergence.

Our results give a cautionary warning to practitioners who apply the optimal bandwidths for linear processes to nonlinear process. In the subsequent sections, where we consider nonparametric regression problems, the assumptions and proofs will be more involved, however the underlying message is the same. That is, more than just the second order autocovariance function may have influence on the rate of convergence, and the rate of convergence can be severely compromised if the usual bandwidths were used.

REMARK 3.2 (Example). It is almost impossible to estimate the 2-mixing size from the observations, in contrast to long memory (c.f. Geweke and Porter-Hudak [1983], Künsch [1987] and Robinson [1995]). However to conclude this section we give an example of a nonlinear process whose 2-mixing size is less than $1 + \delta$, for some $\delta > 0$. Let us consider the ARCH(∞) process (see Robinson [1991]), where $\{Z_t\}$ satisfies

$$Z_t = \sigma_t \varepsilon_t \quad \sigma_t^2 = a_0 + \sum_{j=1}^{\infty} a_j Z_{t-j}^2,$$

with $\mathbb{E}(\varepsilon_t) = 0$ (estimation of ARCH(∞) parameters is considered in Subba Rao [2006]). Giraitis et al. [2000] have shown that if for large t , $a_t \approx t^{-(1+\delta)}$ (for some $\delta < 0$) and $[\mathbb{E}(\varepsilon_t^4)]^{1/2} \sum_{j=1}^{\infty} a_j < 1$, then $|\text{cov}(Z_0^2, Z_t^2)| \approx t^{-(1+\delta)}$. That is, the absolute sum of the covariances is finite, but ‘only just’ if δ is small. Furthermore, if we assume that $|\varepsilon_t| < 1$, then it is straightforward to show that Z_t is a bounded random variable. This means that using the mixing inequality for bounded random variables (see Hall and Heyde [1980]) we

can show

$$|\text{cov}(Z_0^2, Z_t^2)| \leq C \sup_{A \in \sigma(Z_0), B \in \sigma(Z_t)} |P(A \cap B) - P(A)P(B)|,$$

for some $C < \infty$. Altogether this implies that an upper bound for the 2-mixing size of the ARCH(∞) process with $a_t \approx t^{-(1+\delta)}$, is $\mathfrak{v} \leq (1 + \delta)$.

In other words the 2-mixing size for some ARCH(∞) process is small, and far from the geometric rate often assumed in nonparametric estimation. A lower bound for the 2-mixing size can be found in Subba Rao [2007]. \square

4 Nonparametric regression

In this section we consider nonparametric regression, with random design, where the observations are dependent. It is worth mentioning that there has been extensive research done on nonparametric regression with fixed design and dependent errors (c.f. Hall and Hart [1990b], Csörgö and Mielniczuk [1995], and the references therein). In this case typically, one observes Y_t , where $Y_t = \varphi(\frac{t}{T}) + \varepsilon_t$ and $\{\varepsilon_t\}_t$ are stationary random variables with varying degrees of dependence. It has been shown that the rate of convergence depends on the covariance of $\{\varepsilon_t\}_t$, in particular their absolute sum, $\sum_{t=1}^{\infty} |\text{cov}(\varepsilon_0, \varepsilon_t)|$.

In the random design model, one observes the stationary $(1+d)$ -dimensional vector time series $\{(X_t, Z_t)\}_t$, where

$$X_t = \varphi(Z_t) + \varepsilon_t \tag{4.1}$$

with $\mathbb{E}(X_t|Z_t = z) = \varphi(z)$ and $\varepsilon_t = X_t - \mathbb{E}(X_t|Z_t)$. The randomness in this model is determined by two factors: the design $\{Z_t\}$ and the errors $\{\varepsilon_t = X_t - \mathbb{E}(X_t|Z_t)\}$. Therefore, unlike the fixed design model, the rate of convergence of any estimator of φ must depend on the sampling properties of the design density estimator. Thus, it is clear that similar results to those in Section 3 should also apply to an estimator of φ .

We now define the classical Nadaraya-Watson estimator of $\varphi(\cdot)$ and study its sampling properties, under various assumptions on $\{(X_t, Z_t)\}$. Let $p(x, z)$ be the joint density of (X_t, Z_t) . The estimator is

$$\hat{\varphi}(z) = \frac{\hat{g}(z)}{\hat{f}(z)}, \tag{4.2}$$

where $\hat{g}(z) := \frac{1}{T} \sum_{t=1}^T X_t K_b(Z_t - z)$ and $\hat{f}(z) := \frac{1}{T} \sum_{t=1}^T K_b(Z_t - z)$ are estimators of $g(z) = \int x p(x, z) dx$ and $f(z)$, which is the density of Z_t .

We first consider the sampling properties for a particular class of models which satisfy (4.1). Suppose the vector time series $\{(X_t, Z_t)\}$ satisfies the representation

$$X_t = \varphi(Z_t) + h(Z_t)\eta_t \tag{4.3}$$

for some $h : \mathbb{R}^d \rightarrow \mathbb{R}^+$, where the time series $\{Z_t\}$ and $\{\eta_t\}$ are independent of each other. This class of models is similar to the fixed design model $X_t = \varphi(\frac{t}{T}) + \eta_t$, but in (4.3) the design is random and the conditional variance $\text{var}(X_t|Z_t) = h(Z_t)^2 \text{var}(\eta_t)$, depends on the design. This model arises in various applications and we consider one application in Remark 4.2. We will show in the theorem below that the rate of convergence depends both on the mixing size of the design $\{Z_t\}$, but also on the size of the covariances of the process $\{\eta_t\}$ (which we denote by \mathbf{u} , see Definition 3.1).

We require the following assumptions.

ASSUMPTION 4.1 (Densities, moments and kernels). (i) For some $p > 2$ the functions $h^2 \cdot f$ and $|\varphi|^p \cdot f$ are uniformly bounded and we define $q := 1 - 2/p$.

(ii) Let $f^{(t,\tau)}$ and $F^{(t,\tau)}$ be defined as in Assumption 3.1 (ii),

$$g^{(t,\tau)}(z_1, z_2) := \mathbb{E}[X_t X_\tau | Z_t = z_1, Z_\tau = z_2] \cdot f^{(t,\tau)}(z_1, z_2).$$

and $G^{(t,\tau)} := g^{(t,\tau)} - g \otimes g$. Then $\|F^{(t,\tau)}\|_{p_F}$ and $\|G^{(t,\tau)}\|_{p_G}$ are uniformly bounded in t and τ for some $p_F, p_G > 2$. We define $q_F := 1 - 2/p_F$, $q_G := 1 - 2/p_G$ and $q_{FG} := q_F \wedge q_G$.

(iii) The multiplicative kernel K has finite first and p -th moment.

Studying Assumption 4.1(i), we see that it allows for various types of growth of the regression function φ and the conditional variance h . The type of growth depends on the rate the density f decays to zero. For example, if f were the Gaussian density, then exponential growth of φ and h is possible. However, as we shall demonstrate in the theorem below, the larger the p , such that $\sup_x h(x)^p \cdot f(x) < \infty$ and $\sup_x |\varphi(x)|^p \cdot f(x) < \infty$, then the faster the rate of convergence of $|\hat{\varphi}(z) - \varphi(z)|^2$.

THEOREM 4.1. Suppose the stationary time series $\{(X_t, Z_t)\}$ satisfies (4.3), $\{Z_t\}$ is 2-mixing with size \mathbf{v} and the autocovariance of the time series $\{\eta_t\}$ has size \mathbf{u} . Suppose Assumption 4.1 is fulfilled for some $q, q_{FG} \in (0, 1)$.

Let the estimator $\hat{\varphi}(z)$ be defined as in (4.2), where K is a multiplicative kernel of order r . In addition assume that $\varphi \cdot f, f \in \mathfrak{G}_{s,\Delta}^d$ for some $\Delta, s > 0$, f is bounded away from zero and let $\rho = r \wedge s$. Then we have for all $z \in \mathbb{R}^d$

$$|\hat{\varphi}(z) - \varphi(z)|^2 = O_P\left(b^{2\rho} + b^{-d} \cdot T^{-(\mathbf{u} \wedge 1)} + b^{-d(1+q+q_{FG}(1-[(q\mathbf{v}) \vee 1])} \cdot T^{-[(q\mathbf{v}) \wedge 1]}\right), \quad T \rightarrow \infty.$$

REMARK 4.1. We observe that the bound obtained in Theorem 4.1 are similar to the bound derived for the density estimator in Theorem 3.2, where

$$\mathbb{E}|\hat{f}(z) - f(z)|^2 = O\left(b^{2\rho} + b^{-d} \cdot T^{-1} + b^{-d(2-q_F(1-[\mathbf{v} \vee 1])} \cdot T^{-[\mathbf{v} \wedge 1]}\right), \quad (4.4)$$

noting that the result above is for arbitrary dimension d . The difference is the inclusion of the covariance size \mathbf{u} of the errors and the q which ‘balances’ the tails of $1/\varphi$ and f (see Assumption 4.1(i)). However, we observe that we can partition the bound in Theorem 4.1

into three cases, which are similar to the three cases considered in Remark 3.1. Most notably, we observe if $u > 1$ and $v > 1/q_{FG} + 1/q$ then we obtain the usual bound $O(b^{2\rho} + b^{-d} \cdot T^{-1})$ for the MSE.

It is interesting to note that in the case $h^p \cdot f$ and $|\varphi|^p \cdot f$ are uniformly bounded for all p , then $q = 1$ (eg. h and φ are bounded functions and f is exponential density). In this case the bounds given in (4.4) and Theorem 4.1 are quite similar. The main difference is the appearance of q_{FG} rather than q_F and, the term $b^{-d}T^{-(u \wedge 1)}$ which replaces $b^{-d}T^{-1}$. \square

COROLLARY 4.2. *Suppose the assumptions of Theorem 4.1 are satisfied. Let $b^* \approx T^{-\gamma/(2\rho+d)}$ with*

$$\gamma := \begin{cases} \min(u, 1), & qv > 1 + 1/q_{FG}; \\ \min\left(u, [(qv) \wedge 1] \cdot \frac{2\rho+d}{2\rho+d(1+q+q_{FG}(1-[(qv) \vee 1]))}\right), & 1 + 1/q_{FG} \geq qv. \end{cases}, \quad (4.5)$$

then we have $|\hat{\varphi}(z) - \varphi(z)|^2 = O_P\left(T^{-\frac{2\rho}{2\rho+d} \cdot \gamma}\right)$ for all $z \in \mathbb{R}$.

Let us now compare Theorem 4.1 with the bound obtained for the deterministic design $X_t = \varphi(\frac{t}{T}) + \varepsilon_t$, where u is the covariance size of the errors. In the case of the fixed design, the bound for the deviation of the kernel estimator is $O(b^{2\rho} + T^{-(u \wedge 1)}b^{-d})$ (c.f. Hall and Hart [1990b]). We see that the bound in Theorem 4.1 include this term, but also the additional term $O(b^{-d(1+q+q_{FG}(1-[(qv) \vee 1]))} \cdot T^{-[(qv) \wedge 1]})$, which is the influence of the design, in particular, v . If the mixing size of the design were sufficiently large, then the fixed design and random design estimators have the same rate of convergence, $O(T^{-\frac{2\rho}{2\rho+d}})$.

REMARK 4.2 (Example). Examples of processes which satisfy (4.3) are stochastic volatility models (c.f. Linton and Mammen [2004]), where one observes $\{Y_t\}$, which satisfies the representation

$$Y_t = \sigma(Z_t)\eta_t.$$

Here $\{\eta_t\}$ are iid random variables, $\mathbb{E}(\eta_t^2) = 1$ and $\{Z_t\}$ are explanatory variables which can include past values of Y_t . Usually in finance the object is to estimate the conditional volatility σ^2 . By noting that Y_t^2 can be written as

$$Y_t^2 = \sigma(Z_t)^2 + (\eta_t^2 - 1)\sigma(Z_t)^2,$$

we see that Y_t^2 satisfies (4.3) with $X_t = Y_t^2$, $\varepsilon_t = (\eta_t^2 - 1)$ and $h(\cdot) = \sigma(\cdot)^2$. Therefore we can estimate the volatility $\sigma(\cdot)^2$ using (4.2), where $\hat{\sigma}(\cdot)^2$, is the kernel estimator of $\sigma(\cdot)^2$. Furthermore, Theorem 4.1 can be applied to obtain the rate of convergence. More precisely, let v be the mixing size of $\{Z_t\}$, and noting that $\text{cov}\{(\eta_t^2 - 1), (\eta_s^2 - 1)\} = 0$, when $t \neq s$, which implies $u = \infty$, we obtain

$$|\hat{\sigma}(z)^2 - \sigma(z)^2|^2 = O_P\left(b^{2\rho} + b^{-d(1+q+q_{FG}(1-[(qv) \vee 1]))} \cdot T^{-[(qv) \wedge 1]}\right). \quad \square$$

From Corollary 4.2 we see that there are two factors which affect the rate of convergence: the mixing size \mathfrak{v} of the random design $\{Z_t\}$ and the size \mathfrak{u} of the covariance function of $\{\eta_t\}$. There are however several models of interest, which do not satisfy condition (4.3). In this case Theorem 4.1 cannot be applied and it is of interest to investigate what happens in the general case.

Examples of models which do not necessarily satisfy (4.3) include the Cheng-Robinson model, where $\{X_t\}$ satisfies the representation $X_t = F(U_t) + G(U_t, Y_t)$ with $\mathbb{E}(G(U_t, Y_t)|U_t) = 0$ and $\{Y_t\}$ is a long memory process, which is independent of the weakly dependent design random variables $\{U_t\}$ (c.f. Cheng and Robinson [1994], Csörgö and Mielniczuk (1999, 2001)). However, the results are derived under the assumption that $\{Y_t\}$ comes from a linear process and $G(\cdot)$ has a particular form.

An alternative approach is developed in Bosq [1998], who considers nonparametric prediction for time series, where one observes the stationary time series $\{(X_t, Z_t)\}$ and the parameter of interest is $\varphi(z) = \mathbb{E}(X_t|Z_t = z)$. The sampling results in Bosq [1998] are based on the assumption that the mixing size of $\{(X_t, Z_t)\}$ is sufficiently large, (thus excluding Cheng-Robinson type models) yielding an estimate which has the same rate as the kernel estimator for iid random variables.

We now consider the sampling properties of $\hat{\varphi}$, when the observations $\{(X_t, Z_t)\}$ satisfy the general model defined in (4.1), and dependence is quantified through its 2-mixing size, which can be arbitrary.

We will use the following assumptions.

ASSUMPTION 4.2 (Densities, moments and kernels). (i) Let $\mathbb{E}|X_t|^p < \infty$ for some $p > 2$ and define $g^{(p)}(z) := \mathbb{E}[|X_t|^p|Z_t = z] \cdot f(z)$. Then the functions $g^{(p)}$ and f are uniformly bounded and we define $q := 1 - 2/p$.

(ii) Let $f^{(t,\tau)}$ and $F^{(t,\tau)}$ be defined as in Assumption 3.1 (ii) and let $g^{(t,\tau)}$ and $G^{(t,\tau)}$ be defined as in Assumption 3.1 (ii). Then $\|F^{(t,\tau)}\|_{p_F}$ and $\|G^{(t,\tau)}\|_{p_G}$ are uniformly bounded in t and τ for some $p_F, p_G > 2$, where we define $q_F := 1 - 2/p_F$, $q_G := 1 - 2/p_G$ and $q_{FG} := q_F \wedge q_G$.

(iii) The multiplicative kernel K has finite first and p -th moment.

We note that assumptions above are similar to Assumption 4.1. The difference lies in Assumption 4.1(i) and Assumption 4.2(i). Assumption 4.2(i) is in terms of moments whereas Assumption 4.1(i) is in terms of functions.

In the following theorem we derive an error bound for the estimator $\hat{\varphi}$.

THEOREM 4.3. Suppose the stationary time series $\{(X_t, Z_t)\}$ satisfies (4.1), and is 2-mixing of size \mathfrak{v} . Furthermore, Assumption 4.2 is fulfilled for some $q_{FG}, q \in (0, 1)$.

Let the estimator $\hat{\varphi}(z)$ be defined as in (4.2), where K is a multiplicative kernel of order r . In addition assume that $\varphi \cdot f, f \in \mathfrak{G}_{s,\Delta}^d$ for some $\Delta, s > 0$, f is bounded away from zero

and let $\rho = r \wedge s$. Then we have for all $z \in \mathbb{R}^d$

$$|\hat{\varphi}(z) - \varphi(z)|^2 = O_P\left(b^{2\rho} + b^{-d} \cdot T^{-1} + b^{-d(1+q+q_{FG}(1-[(q\mathbf{v}) \vee 1]) \cdot T^{-[(q\mathbf{v}) \wedge 1]})}\right), \quad T \rightarrow \infty.$$

We now obtain the rates of convergence using the optimal bandwidth.

COROLLARY 4.4. *Suppose the assumptions in Theorem 4.3 are satisfied. Let $b^* \approx T^{-\gamma/(2\rho+d)}$ with*

$$\gamma := \begin{cases} 1, & q\mathbf{v} > 1 + q/q_{FG}; \\ [(q\mathbf{v}) \wedge 1] \cdot \frac{2\rho+d}{2\rho+d(1+q+q_{FG}(1-[(q\mathbf{v}) \vee 1])}), & 1 + q/q_{FG} \geq q\mathbf{v}, \end{cases} \quad (4.6)$$

then we have $|\hat{\varphi}(z) - \varphi(z)|^2 = O_P\left(T^{-\frac{2\rho}{2\rho+d} \cdot \gamma}\right)$ for all $z \in \mathbb{R}$.

5 Nonparametric panel time series

In recent years, panel time series have often been used to model the relationship and dynamics between several observed time series. Typically we let $X_{t,i}$ denote the observation of the i th individual at time t , where $i = 1, \dots, N$ and $t = 1, \dots, T$. We also assume that we observe some explanatory variables $Z_{t,i}$ which are known to influence $X_{t,i}$. Several models have been proposed to model the complex relationship between individuals, ranging from parametric models (c.f. Baltagi [2001], Hjellvik and Tjøstheim [1999], Dahlhaus and Feiler [2005], and the references therein) to nonparametric additive models (c.f. Mammen et al. [2005]). In this section, we take the nonparametric route, and use the methods developed in the sections above to obtain an estimator of the mean function and study its sampling properties. The results in this section can be used in various applications, an interesting example is the estimation of the covariance function of spatial-temporal models considered in Johannes et al. [2007].

Let us suppose the affect of the explanatory variables is common over all individuals. To be precise, the response and explanatory variables $\{(X_{t,i}, Z_{t,i})\}_t$ form a $(1+d)$ -dimensional stationary vector time series which satisfies the relation

$$\mathbb{E}[X_{t,i}|Z_{t,i} = z] = \varphi(z) \quad \forall z \in \mathbb{R}^d, i \in \mathbb{N}, t \in \mathbb{Z}. \quad (5.1)$$

We describe the dependence of $\{(X_{t,i}, Z_{t,i})\}$, by assuming it is 2-mixing over time, see Definition 5.1, below. We note that the model considered in Hjellvik et al. [2004] and Mammen et al. [2005], can be used as a particular example of (5.1).

We now define an estimator for φ . Note that we do not suppose that different individuals, say $(X_{t,i}, Z_{t,i})$ and $(X_{t,j}, Z_{t,j})$ are identically distributed (have common densities). Let $f_i(x, z)$ denote the joint density of the random vector $(X_{t,i}, Z_{t,i})$ for $i \in \mathbb{N}$. Moreover, let $f_i(z)$ denote the marginal density of $Z_{t,i}$. Using these densities we can rewrite (5.1) as

$$\varphi(z) = \mathbb{E}[X_{t,i}|Z_{t,i} = z] = \int x \frac{f_i(x, z)}{f_i(z)} dx =: \frac{g_i(z)}{f_i(z)}, \quad \forall z \in \mathbb{R}^d, t \in \mathbb{Z}, i \in \mathbb{N}.$$

Furthermore, using the above, it is easily verified that

$$\varphi(z) = \frac{\frac{1}{N} \sum_{i=1}^N g_i(z)}{\frac{1}{N} \sum_{i=1}^N f_i(z)}, \quad \forall z \in \mathbb{R}^d, N \in \mathbb{N} \quad (5.2)$$

which motivates the following estimator of φ .

Given the observations $\{(X_{t,i}, Z_{t,i}); t = 1, \dots, T; i = 1, \dots, N\}$ our object is to estimate φ and consider its sampling properties. The identity (5.2) suggests as an estimator of $\varphi(z)$

$$\hat{\varphi}(z) = \frac{\frac{1}{N} \sum_{i=1}^N \hat{g}_i(z)}{\frac{1}{N} \sum_{i=1}^N \hat{f}_i(z)}, \quad (5.3)$$

using for each $i = 1, \dots, N$

$$\hat{g}_i(z) := \frac{1}{T} \sum_{t=1}^T X_{t,i} K_{b_i}(Z_{t,i} - z) \quad \text{and} \quad \hat{f}_i(z) := \frac{1}{T} \sum_{t=1}^T K_{b_i}(Z_{t,i} - z) \quad (5.4)$$

as estimators of $g_i(z)$ and $f_i(z)$, respectively.

We quantify the dependence both over time and between individuals through their 2-mixing rates.

DEFINITION 5.1. *The panel time series $\{(X_{t,i}, Z_{t,i})\}_t$, $i \in \mathbb{N}$, is said to be 2-mixing with size \mathfrak{v} and \mathfrak{u} , if for all $i, j \in \mathbb{N}$*

$$\sup_{A \in \sigma(X_{t,i}, Z_{t,i}), B \in \sigma(X_{\tau,j}, Z_{\tau,j})} |P(A \cap B) - P(A)P(B)| \leq C \begin{cases} |t - \tau|^{-\mathfrak{v}}; & \text{if } i = j, \\ |t - \tau|^{-\mathfrak{u}}; & \text{otherwise,} \end{cases}$$

for some $C < \infty$ independent of i, j, t and τ .

In the results below we will show that the rate of convergence of $\varphi(\cdot)$ is determined by the smallest mixing size $(\mathfrak{v} \wedge \mathfrak{u})$. However in the case that $\mathfrak{v} < \mathfrak{u}$ and the number of individuals N grow with T , the rate is determined, solely, by \mathfrak{u} .

REMARK 5.1. We note that in Definition 5.1 we have two 2-mixing sizes, the size \mathfrak{v} describes the dependence of the time series $\{(X_{t,i}, Z_{t,i})\}_t$, whereas the size \mathfrak{u} describes the dependence between individuals over time. By separating these two sizes we can model different behaviours. A simple example is $X_{t,i} = \varphi(Z_i) + \varepsilon_{t,i}$, where for a given individual i , the explanatory variable Z_i is fixed over time, $\{\varepsilon_{t,i}\}$ and $\{Z_i\}$ are iid random variables. In this example, $\mathfrak{v} = 0$ and $\mathfrak{u} = \infty$. \square

To obtain the sampling properties of $\hat{\varphi}$ we require the following assumptions, which are an extension of Assumption 4.2 to panel data.

ASSUMPTION 5.1 (Densities, moments and kernels). *(i) For all $i \in \mathbb{N}$ let $\mathbb{E}[|X_{t,i}|^p] < \infty$ for some $p > 2$ and define $g_i^{(p)}(z) := \mathbb{E}[X_{t,i}^p | Z_{t,i} = z] \cdot f_i(z)$. Then the functions $g_i^{(p)}$ and f_i , for all $i \in \mathbb{N}$, are uniformly bounded and we define $q := 1 - 2/p$.*

(ii) For each $t, \tau \in \mathbb{Z}$ and $i, j \in \mathbb{N}$ let $f_{i,j}^{(t,\tau)}$ denote the joint density of $(Z_{t,i}, Z_{\tau,j})$ and let $g_{i,j}^{(t,\tau)}(z_1, z_2) := \mathbb{E}[X_{t,i}X_{\tau,j} | Z_{t,i} = z_1, Z_{\tau,j} = z_2] \cdot f_{i,j}^{(t,\tau)}(z_1, z_2)$. Define $F_{i,j}^{(t,\tau)} := f_{i,j}^{(t,\tau)} - f_i \otimes f_j$ and $G_{i,j}^{(t,\tau)} := g_{i,j}^{(t,\tau)} - g_i \otimes g_j$. Then $\|F_{i,j}^{(t,\tau)}\|_{p_F}$ and $\|G_{i,j}^{(t,\tau)}\|_{p_G}$ are uniformly bounded in i, j, t and τ for some $p_F, p_G > 2$. We define $q_F = 1 - 2/p_F$, $q_G = 1 - 2/p_G$ and $q_{FG} := q_F \wedge q_G$.

(iii) The multiplicative kernel K has a finite first and p -th moment.

We now obtain a bound for the deviation of $\hat{\varphi}(z)$, as N is kept fixed and $T \rightarrow \infty$.

THEOREM 5.1. *Let us suppose that the stationary panel time series $\{(X_{t,i}, Z_{t,i})\}$ satisfies (5.1), for all $i, j \in \mathbb{N}$, and is 2-mixing with size \mathbf{v} and \mathbf{u} (as defined in Definition 5.1). Suppose Assumption 5.1 is fulfilled for some $q, q_{FG} \in (0, 1)$.*

Let the estimator $\hat{\varphi}$ be defined in (5.3), where for each $i = 1, \dots, N$ the nonparametric estimators \hat{g}_i and \hat{f}_i given in (5.4) are constructed using a multiplicative kernel K of order $r > 0$. In addition assume for each $i = 1, \dots, N$, that the functions $\varphi \cdot f_i$ and f_i belong to $\mathfrak{G}_{s_i, \Delta}^d$ for $s_i, \Delta > 0$, f_i is bounded away from zero and let $\rho_i = r \wedge s_i$. Then we have for all $z \in \mathbb{R}^d$

$$|\hat{\varphi}(z) - \varphi(z)|^2 = O_p\left(\frac{1}{N} \sum_{i=1}^N \left\{ b_i^{2\rho_i} + T^{-1} \cdot b_i^{-d} + T^{-[(qu) \wedge 1]} \cdot b_i^{-d(1+q+q_{FG}-q_{FG}[(qu) \vee 1])} \right. \right. \\ \left. \left. + N^{-1} \cdot T^{-[(qv) \wedge 1]} \cdot b_i^{-d(1+q+q_{FG}-q_{FG}[(qv) \vee 1])} \right\} \right), \quad T \rightarrow \infty. \quad (5.5)$$

Comparing Theorem 4.3 with the theorem above, we see, besides the summation $\frac{1}{N} \sum_{i=1}^N$, the addition of an extra term $T^{-[(qu) \wedge 1]} \cdot b_i^{-d(1+q+q_{FG}-q_{FG}[(qu) \vee 1])}$ due to the dependence between individuals over time. Altogether this implies that the bound can be partitioned into nine different cases, depending on the values of \mathbf{u} and \mathbf{v} (compare this with the bound in Theorem 4.3, which can be partitioned into three cases). However, the nine different bounds can be grouped into two main cases; when $\mathbf{u} \leq \mathbf{v}$ and $\mathbf{u} > \mathbf{v}$, we consider these two cases in the corollaries below.

If $\mathbf{u} \leq \mathbf{v}$, we notice that the third term dominates the fourth term, in other words there is a larger dependence between individuals over time than for each individual over time. We consider this case below.

COROLLARY 5.2. *Suppose the assumptions in Theorem 5.1 are satisfied and $\mathbf{u} \leq \mathbf{v}$. For each $i = 1, \dots, N$, let $b_i^* \approx T^{-\gamma_i/(2\rho_i+d)}$ with*

$$\gamma_i := \begin{cases} 1, & qu > 1 + q/q_{FG}; \\ [qu \wedge 1] \cdot \frac{2\rho_i+d}{2\rho_i+d(1+q+q_{FG}(1-[(qu) \vee 1]))}, & 1 + q/q_{FG} \geq qu, \end{cases} \quad (5.6)$$

then for all $z \in \mathbb{R}^d$ we have $|\hat{\varphi}(z) - \varphi(z)|^2 = O_P\left(\frac{1}{N} \sum_{i=1}^N T^{-\frac{2\rho_i}{2\rho_i+d} \cdot \gamma_i}\right)$, $T \rightarrow \infty$.

Studying the corollary above we see that the rate of convergence is determined by \mathbf{u} . In other words, there is no benefit in the estimation by including several individuals N .

Furthermore we see that the ‘usual’ nonparametric rate of convergence is only achieved if $\mathbf{u} > 1/q + 1_{FG}$.

Let us now consider the situation where $\mathbf{v} < \mathbf{u}$, that is there is less dependence between two different individuals over time than one individual observed over time (see Remark 5.1 for an example). This scenario is more likely to arise in real applications. In this case, the fourth term dominates the third term in (5.5), which suggests that increasing the number of individuals does yield a faster rate of convergence. We notice that the usual nonparametric rate of convergence can only be obtained if $\mathbf{u} > 1/q_{FG} + 1/q$.

COROLLARY 5.3. *Suppose the assumptions of Theorem 5.1 are satisfied and $\mathbf{v} \leq \mathbf{u}$. For each $i = 1, \dots, N$, let $b_i^* \approx N^{-\zeta_i/(2\rho_i+d)} \cdot T^{-\delta_i/(2\rho_i+d)}$ where $\zeta_i \geq 0$ and*

$$\delta_i := \begin{cases} 1, & q\mathbf{v} > 1 + q/q_{FG}; \\ [q\mathbf{v} \wedge 1] \frac{2\rho_i+d}{2\rho_i+d(1+q+q_{FG}(1-[q\mathbf{v} \wedge 1]))}, & 1 + q/q_{FG} \geq q\mathbf{v}. \end{cases} \quad (5.7)$$

Then for all $z \in \mathbb{R}^d$ we have

$$|\hat{\varphi}(z) - \varphi(z)|^2 = O_P\left(\frac{1}{N} \sum_{i=1}^N \left\{ N^{-\frac{2\rho_i}{2\rho_i+d} \cdot \zeta_i} \cdot T^{-\frac{2\rho_i}{2\rho_i+d} \cdot \delta_i} \cdot [N^{\frac{\zeta_i}{\gamma_i} \cdot [(q\mathbf{u}) \wedge 1]} \cdot T^{-\frac{\gamma_i - \delta_i}{\delta_i} \cdot [(q\mathbf{u}) \wedge 1]} + N^{-1 + \frac{\zeta_i}{\delta_i} \cdot [(q\mathbf{v}) \wedge 1]} \right\}\right), \quad T \rightarrow \infty, \quad (5.8)$$

where γ_i is defined in (5.6).

Studying the corollary above, we see if N is kept fixed, then the terms inside the inner bracket of (5.8) are of order $O_P(1)$, therefore the rate of convergence is $O_P\left(\frac{1}{N} \sum_{i=1}^N T^{-\frac{2\rho_i}{2\rho_i+d} \cdot \delta_i}\right)$. Thus, combining Corollaries 5.2 and 5.3 we have for arbitrary \mathbf{v} and \mathbf{u} the rate $O_P\left(\frac{1}{N} \sum_{i=1}^N T^{-\frac{2\rho_i}{2\rho_i+d} \cdot [\gamma_i \wedge \delta_i]}\right)$. Therefore, we obtain the usual nonparametric rate if $(\mathbf{v} \wedge \mathbf{u}) > 1/q_{FG} + 1/q$.

Altogether the corollaries above imply that the rate of convergence depends on the slowest mixing rate, within or between the individuals. Let us suppose that $\mathbf{v} < \mathbf{u}$, if we closely examine (5.8) we see if ζ_i is chosen such that $\zeta_i < \gamma_i[(q\mathbf{u}) \wedge 1]$, $\gamma_i < \delta_i[(q\mathbf{u}) \wedge 1]$ (noting that the former inequality implies the later, since $\mathbf{v} < \mathbf{u}$) and $(\delta_i - \gamma_i) > 0$ (which is the case when $\mathbf{v} < \mathbf{u}$), then the terms inside the inner bracket of (5.8) become small for large N . This means that increasing the number of individuals leads to a faster rate of convergence. We show in the corollary below if we allow the number of individuals N to grow as T grows, then the rate of convergence will depend only on \mathbf{u} and no longer on the smaller \mathbf{v} (unlike the case that N is fixed).

COROLLARY 5.4. *Suppose the assumptions in Theorem 5.1 are satisfied and let $\mathbf{v} \leq \mathbf{u}$. Furthermore assume there exists a $\zeta_i > 0$ such that for each $i \in \mathbb{N}$, $N^{\zeta_i} \approx T^{(\gamma_i - \delta_i)}$, where γ_i and δ_i are defined in (5.6) and (5.7), respectively and $\delta_i/[(q\mathbf{v}) \wedge 1] \geq \zeta_i$. Then given $b_i^* \approx N^{-\zeta_i/(2\rho_i+d)} \cdot T^{-\delta_i/(2\rho_i+d)} = T^{-\gamma_i/(2\rho_i+d)}$ we have for all $z \in \mathbb{R}^d$*

$$|\hat{\varphi}(z) - \varphi(z)|^2 = O_P\left(\frac{1}{N} \sum_{i=1}^N T^{-\frac{2\rho_i}{2\rho_i+d} \cdot \gamma_i}\right), \quad T \rightarrow \infty. \quad (5.9)$$

We see from the corollary above if the number of individuals, N , grows at the rate $N \approx T^{\frac{\gamma_i - \delta_i}{\zeta_i}}$, where ζ_i cannot be too large, in particular $\delta_i[(q\mathfrak{v}) \wedge 1] \geq \zeta_i$, then the rate of convergence depends only on the mixing size \mathfrak{u} (compare this with Corollaries 5.2 and 5.3, where the mixing size depends on $(\mathfrak{v} \wedge \mathfrak{u})$). Furthermore if $q\mathfrak{u} \geq q/q_{FG} + 1$ then we have the usual nonparametric rate $|\hat{\varphi}(z) - \varphi(z)|^2 = O_P\left(\frac{1}{N} \sum_{i=1}^N T^{-\frac{2\rho_i}{2\rho_i+d}}\right)$. In the special case that φ and all the marginal densities f_i belong to the same smoothness class, that is for all i , $\rho = \rho_i$, then (5.9) simplifies to $|\hat{\varphi}(z) - \varphi(z)|^2 = O_P\left(T^{-\frac{2\rho}{2\rho+d}}\right)$.

REMARK 5.2. It is worth mentioning that similar results to those in Theorem 5.1 and Corollaries 5.2, 5.3 and 5.4 can be obtained for the model (4.3) considered in Section 4. \square

6 Discussion

In this paper we have considered nonparametric estimation for dependent data. Focusing on the case that the observations are nonlinear and highly dependent. We have obtained bounds for the kernel density estimator and also rates of convergence of two types of nonparametric regression models, both using the 2-mixing dependence measure. We show that when the assumption of linearity is relaxed, the rate of convergence does not necessarily depend on the autocovariance function of the observations. We demonstrate that 2-mixing is a natural measure of dependence for panel data and obtained rates of convergence for the common mean function in panel time series.

As we are working under relatively weak conditions, we do not claim that the bounds obtained are minimax. However, the bounds can be considered as the worst case scenario for the nonparametric estimator. In future work, it would be of interest to investigate if the bounds in the paper are indeed close to minimax for certain nonlinear time series. In this paper we have derived bounds for the estimator using the optimal bandwidth. However the optimal bandwidth is constructed under the assumption that the 2-mixing size is known. It would also be of interest to develop bandwidth selection methods when the 2-mixing size of the observations is unknown.

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A Appendix: Proofs

A.1 Proofs: Nonparametric density estimation

We now prove the results in Section 3. We mention that Theorem 3.2 is stated for an univariate time series $\{Z_t\}$, however the proofs of the results in Sections 4 and 5 require

results in the multivariate case. Therefore to save space, we give the proof of Theorem 3.2 for a d -dimensional vector time series $\{Z_t\}$.

LEMMA A.1. *Suppose the time series $\{Z_t\}$ is 2-mixing with size \mathfrak{v} and Assumption 3.1 is fulfilled for some $q_F \in (0, 1)$. If $1 \leq t, \tau \leq T$, then ²*

$$|\text{cov}\{K_b(Z_t - z), K_b(Z_\tau - z)\}| \lesssim \min\left(b^{-d(1-q_F)}; b^{-2d}|t - \tau|^{-\mathfrak{v}}\right). \quad (\text{A.1})$$

PROOF. Writing the covariance as an integral, and using the notation in Assumption 3.1 (ii) we have

$$\text{cov}\{K_b(Z_t - z), K_b(Z_\tau - z)\} = \int K_b(u - z)K_b(v - z)F^{(t,\tau)}(u, v)dudv.$$

Now by using Hölder's inequality with $p_F^{-1} + \bar{p}_F^{-1} = 1$, and recalling that $q_F = 1 - 2/p_F$, it is clear that

$$|\text{cov}\{K_b(Z_t - z), K_b(Z_\tau - z)\}| \leq \frac{1}{b^{2d}} \cdot b^{2d/\bar{p}_F} \|K\|_{\bar{p}_F}^2 \cdot \|F^{t,\tau}\|_{p_F} \lesssim b^{-d(1-q_F)}.$$

Using Assumption 3.1 we have that $\|F^{t,\tau}\|_{p_F}$ is uniformly bounded and by using Lyapounov's inequality $\|K\|_{\bar{p}_F} < \infty$ for all $1 < \bar{p}_F < 2$. This gives us the first bound in (A.1). On the other hand, under Assumption 3.1 (i) the kernel K is uniformly bounded and therefore, using the 2-mixing property of $\{Z_t\}$ together with Hall and Heyde [1980], Theorem A.5, we obtain

$$|\text{cov}\{K_b(Z_t - z), K_b(Z_\tau - z)\}| \lesssim b^{-2d} \cdot |t - \tau|^{-\mathfrak{v}},$$

which gives the second bound in (A.1). \square

PROOF OF THEOREM 3.2. We mention that parts of the following proof are motivated by techniques used in Bosq [1998], where nonparametric smoothing was considered for univariate time series. Consider the standard variance bias decomposition

$$\mathbb{E}|\hat{f}(z) - f(z)|^2 = \text{var}(\hat{f}(z)) + |\mathbb{E}\hat{f}(z) - f(z)|^2. \quad (\text{A.2})$$

Under the stated assumptions we will derive the following two bounds, which give together the result of the theorem. The bias is bounded by

$$|\mathbb{E}\hat{f}(z) - f(z)|^2 \lesssim b^{2\rho}, \quad (\text{A.3})$$

while for the variance we have

$$\text{var}(\hat{f}(z)) \lesssim T^{-1} \cdot b^{-d} + T^{-[\mathfrak{v} \wedge 1]} \cdot b^{-d(2+q_F-q_F[\mathfrak{v} \vee 1])}. \quad (\text{A.4})$$

²We write $A \lesssim B$ if there exists a positive constant c such that $A \leq cB$.

Proof of (A.3). We can write

$$\mathbb{E}\hat{f}(z) = \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left(K_b(Z_t - z) \right) = \int du f(u) K_b(u - z).$$

Since $f \in \mathfrak{G}_{s,\Delta}^d$ and K is a multiplicative kernel of order r with $\int du |u|^r K(u) \leq S_K$, using a Taylor expansion up to the order $\rho = \min(r, s)$ leads to $\mathbb{E}\hat{f}(z) = f(z) + b^\rho R$ with reminder $|R| \leq \Delta S_K < \infty$, which proves (A.3).

In order to proof (A.4), we consider the expansion

$$\begin{aligned} \text{var}(\hat{f}(z)) &= \frac{1}{T^2} \sum_{t=1}^T \text{var} \{K_b(Z_t - z)\} + \frac{2}{T^2} \sum_{t>\tau} \text{cov} \{K_b(Z_t - z), K_b(Z_\tau - z)\} \\ &=: A_1 + A_2. \end{aligned} \tag{A.5}$$

We will show that $|A_1| \lesssim T^{-1} \cdot b^{-d}$ and

$$|A_2| \lesssim \begin{cases} T^{-\mathfrak{v}} \cdot b^{-2d}, & \mathfrak{v} \leq 1; \\ T^{-1} \cdot \{b^{-d} + b^{-d(2+q_F-q_F\mathfrak{v})}\}, & 1 < \mathfrak{v}. \end{cases} \tag{A.6}$$

Furthermore, if $0 \leq \mathfrak{v} \leq 1/q_F + 1$ then $|A_1|$ is dominated by $|A_2|$. Whereas for $\mathfrak{v} > 1/q_F + 1$ the terms $|A_1|$ and $|A_2|$ are of the same order $O(T^{-1}b^{-d})$. Therefore, the bounds derived for $|A_2|$ will lead to (A.4).

First let us consider A_1 . Due to stationarity, we have the bound

$$T \cdot A_1 \leq \mathbb{E}[K_b^2(Z_1 - z)] = \int du f(u) K_b^2(u - z).$$

Since under the stated assumptions $\|K\|_2 < \infty$ and the density f is uniformly bounded this leads to $A_1 \lesssim T^{-1} \cdot b^{-d}$.

The term $T \cdot |A_2|$ is bounded by the sum $4 \sum_{t=2}^T |\text{cov} \{K_b(Z_t - z), K_b(Z_1 - z)\}|$. If $\mathfrak{v} \leq 1$ then we estimate the sum using the second bound in Lemma A.1, i.e., $T \cdot |A_2| \lesssim T^{-\mathfrak{v}+1} b^{-2d}$, which is the first bound in (A.6). On the other hand if $\mathfrak{v} > 1$ we partition the sum into two parts which we estimate separately using the bounds in Lemma A.1, thus giving us

$$T \cdot |A_2| \lesssim \left\{ \sum_{t=2}^h b^{-d(1-q_F)} + \sum_{t=h+1}^T b^{-2d} t^{-\mathfrak{v}} \right\} \lesssim \left\{ h \cdot b^{-d(1-q_F)} + h^{-\mathfrak{v}+1} \cdot b^{-2d} \right\}.$$

Thereby using $h \approx b^{-dq_F}$ we obtain $T \cdot |A_2| \lesssim b^{-d} + b^{-d(2+q_F-q_F\mathfrak{v})}$, i.e., the second bound in (A.6). Thus we have proved (A.4). \square

PROOF OF COROLLARY 3.3 Under the assumption on the bandwidth the result is obtained by balancing the terms in the bound given in Theorem 4.1. \square

A.2 Proofs: Nonparametric regression

We now prove the results in Section 4.

LEMMA A.2. *Suppose the stationary time series $\{X_t, Z_t\}$ satisfies (4.3), and $\{Z_t\}$ is 2-mixing with size \mathfrak{v} and the autocovariances of the time series $\{\eta_t\}$ have size \mathfrak{u} (see Definition 3.1). Suppose Assumption 4.1 is fulfilled for some $q, q_G \in (0, 1)$. If $1 \leq t, \tau \leq T$, then*

$$|\text{cov}\{X_t K_b(Z_t - z), X_\tau K_b(Z_\tau - z)\}| \lesssim b^{-d(1-q_G)}, \quad (\text{A.7})$$

$$|\text{cov}\{\varphi(Z_t) K_b(Z_t - z), \varphi(Z_\tau) K_b(Z_\tau - z)\}| \lesssim b^{-d(1+q)} |t - \tau|^{-q\mathfrak{v}}, \quad (\text{A.8})$$

$$|\text{cov}\{h(Z_t) K_b(Z_t - z) \eta_t, h(Z_\tau) K_b(Z_\tau - z) \eta_\tau\}| \lesssim b^{-d} |t - \tau|^{-\mathfrak{u}}. \quad (\text{A.9})$$

PROOF. Using the notation in Assumption 4.1 together with Hölder's inequality, and recalling that $q_G = 1 - 2/p_G$ with $p_G^{-1} + \bar{p}_G^{-1} = 1$, we have

$$|\text{cov}\{X_t K_b(Z_t - z), X_\tau K_b(Z_\tau - z)\}| \lesssim b^{-d(1-q_G)},$$

where we use that under Assumption 4.1, K has finite $1 < \bar{p}_G < p$ moment (by Lyapounov's inequality) and $\|G_{t,\tau}\|_{p_G}$ is uniformly bounded. This gives us (A.7).

We now prove (A.8). Under Assumption 4.1 the function $|\varphi|^p \cdot f$ is uniformly bounded and $\|K\|_p$ is finite for some $p = 2/(1-q) > 2$, therefore we have $[\mathbb{E}|\varphi(Z_1) K_b(Z_1 - z)|^p]^{2/p} \lesssim b^{-d(q+1)}$. Using the 2-mixing property of $\{Z_t\}$ together with Hall and Heyde [1980], Theorem A.6, we obtain (A.8).

We now prove (A.9). The series $\{Z_t\}$ and $\{\eta_t\}$ are independent, therefore expanding the term $A := \text{cov}\{h(Z_t) K_b(Z_t - z) \eta_t, h(Z_\tau) K_b(Z_\tau - z) \eta_\tau\}$ gives

$$A = \text{cov}(\eta_t, \eta_\tau) \cdot \mathbb{E}[h(Z_t) K_b(Z_t - z) h(Z_\tau) K_b(Z_\tau - z)].$$

Since the covariance of the time series $\{\eta_t\}$ has size \mathfrak{u} , applying the Cauchy-Schwarz inequality gives

$$|A| \lesssim |t - \tau|^{-\mathfrak{u}} \cdot \mathbb{E}|h(Z_1) K_b(Z_1 - z)|^2.$$

Under Assumption 4.1 the function $|h|^2 \cdot f$ is uniformly bounded and $\|K\|_2 < \infty$, therefore $\mathbb{E}|h(Z_1) K_b(Z_1 - z)|^2 \lesssim b^{-d}$, and hence we obtain (A.9). \square

LEMMA A.3. *Suppose the stationary time series $\{Z_t\}$ is 2-mixing with size \mathfrak{v} and Assumption 4.1 is fulfilled for some $q, q_F \in (0, 1)$. If $1 \leq t, \tau \leq T$, then*

$$|\text{cov}\{K_b(Z_t - z), K_b(Z_\tau - z)\}| \lesssim \min\left(b^{-d(1-q_F)}; b^{-d(1+q)} |t - \tau|^{-q\mathfrak{v}}\right). \quad (\text{A.10})$$

PROOF. The proof is very similar to the proof of Lemma A.2 and we omit the details. \square

LEMMA A.4. *Suppose the assumptions in Theorem 4.1 are satisfied. Let \hat{g} be defined as in (4.2). Then we have*

$$\mathbb{E}|\hat{g}(z) - g(z)| \lesssim b^{2\rho} + b^{-d}T^{-1} + b^{-d(1+q+q_G(1-[(q\mathbf{v})\vee 1]))}T^{-[(q\mathbf{v})\wedge 1]} + b^{-d}T^{-(u\wedge 1)}. \quad (\text{A.11})$$

PROOF. Consider the standard variance bias decomposition

$$\mathbb{E}|\hat{g}(z) - g(z)|^2 = \text{var}(\hat{g}(z)) + |\mathbb{E}\hat{g}(z) - g(z)|^2. \quad (\text{A.12})$$

Under the stated assumptions we will derive the following two bounds, which altogether give the estimate in (A.11). The bias is bounded by

$$|\mathbb{E}\hat{g}(z) - g(z)|^2 \lesssim b^{2\rho}, \quad (\text{A.13})$$

while for the variance we have

$$\text{var}(\hat{g}(z)) \lesssim b^{-d}T^{-1} + b^{-d(1+q+q_G(1-[(q\mathbf{v})\vee 1]))}T^{-[(q\mathbf{v})\wedge 1]} + b^{-d}T^{-(u\wedge 1)}. \quad (\text{A.14})$$

We first prove (A.13). We can write

$$\mathbb{E}\hat{g}(z) = \frac{1}{T} \sum_{t=1}^T \mathbb{E}(\mathbb{E}[X_t|Z_t]K_b(Z_t - z)) = \int du g(u)K_b(u - z).$$

Since $g \in \mathfrak{G}_{s,\Delta}^d$ and K is a multiplicative kernel of order r with $\int du |u|^r K(u) \leq S_K$, using a Taylor expansion up to the order $\rho = \min(r, s)$ leads to $\mathbb{E}\hat{g}(z) = g(z) + b^\rho R$ with reminder $|R| \leq \Delta S_K < \infty$, which proves (A.13).

In order to proof (A.14), we consider the expansion

$$\begin{aligned} \text{var}(\hat{g}(z)) &= \frac{1}{T^2} \sum_{t=1}^T \text{var}\{X_t K_b(Z_t - z)\} + \frac{2}{T^2} \sum_{t>\tau} \text{cov}\{X_t K_b(Z_t - z), X_\tau K_b(Z_\tau - z)\} \\ &=: A_1 + A_2. \end{aligned} \quad (\text{A.15})$$

We will show that $|A_1| \lesssim T^{-1} \cdot b^{-d} + T^{-(u\wedge 1)} \cdot b^{-d}$ and

$$|A_2| \lesssim \begin{cases} T^{-q\mathbf{v}} \cdot b^{-d(1+q)} + T^{-(u\wedge 1)} \cdot b^{-d}, & q\mathbf{v} \leq 1; \\ T^{-1} \cdot b^{-d} + T^{-1} \cdot b^{-d(1+q+q_G(1-q\mathbf{v}))} + T^{-(u\wedge 1)} \cdot b^{-d}, & 1 < q\mathbf{v}. \end{cases} \quad (\text{A.16})$$

Furthermore, if $0 \leq q\mathbf{v} \leq q/q_G + 1$ then we show that $|A_1|$ is dominated by $|A_2|$. Whereas for $q\mathbf{v} > q/q_G + 1$ the terms $|A_1|$ and $|A_2|$ are of the same order $O(T^{-1} \cdot b^{-d} + T^{-(u\wedge 1)} \cdot b^{-d})$. Therefore, the bounds derived for $|A_2|$ will lead to the estimates in (A.14).

First let us consider A_1 . Due to stationarity of the process, we have the bound

$$T \cdot A_1 \leq \mathbb{E}|X_1 K_b(Z_1 - z)|^2 \lesssim \mathbb{E}|\varphi(Z_1)K_b(Z_1 - z)|^2 + \mathbb{E}|h(Z_1)K_b(Z_1 - z)|^2.$$

Under the stated assumptions the functions $|\varphi|^p \cdot f$ with $p > 2$ and $|h|^2 \cdot f$ are uniformly bounded and the kernel $\|K\|_2 < \infty$, therefore $A_1 \lesssim T^{-1} \cdot b^{-d}$.

Let us now consider the term A_2 , which is bound by

$$T \cdot |A_2| \leq 4 \sum_{t=2}^T |\text{cov} \{X_t K_b(Z_t - z), X_1 K_b(Z_1 - z)\}|, \quad (\text{A.17})$$

where using representation (4.3) the t -th summand in (A.17) can be estimated by

$$|\text{cov} \{\varphi(Z_t) K_b(Z_t - z), \varphi(Z_1) K_b(Z_1 - z)\}| \\ + |\text{cov} \{h(Z_t) K_b(Z_t - z) \eta_t, h(Z_1) K_b(Z_1 - z) \eta_1\}|. \quad (\text{A.18})$$

If $q\mathfrak{v} \leq 1$ and $\mathfrak{u} \leq 1$ then we bound the sum (A.17) using (A.18), i.e.,

$$T \cdot |A_2| \lesssim \sum_{t=2}^T |\text{cov} \{\varphi(Z_t) K_b(Z_t - z), \varphi(Z_1) K_b(Z_1 - z)\}| \\ + \sum_{t=2}^T |\text{cov} \{h(Z_t) K_b(Z_t - z) \eta_t, h(Z_1) K_b(Z_1 - z) \eta_1\}|. \quad (\text{A.19})$$

We use the bounds (A.8) and (A.9) in Lemma A.2 to estimate each of the sums in (A.19) separately, which gives

$$T \cdot |A_2| \lesssim T^{-q\mathfrak{v}+1} \cdot b^{-d(1+q)} + T^{-\mathfrak{u}+1} \cdot b^{-d}. \quad (\text{A.20})$$

On the other hand if $q\mathfrak{v} > 1$ or if $\mathfrak{u} > 1$ we partition the sum (A.17) into two parts, where we estimate the first part using the bound (A.7) in Lemma A.2 and the second using (A.18), thus giving us

$$T \cdot |A_2| \lesssim h \cdot b^{-d(1-q_G)} + \sum_{t=h+1}^T |\text{cov} \{\varphi(Z_t) K_b(Z_t - z), \varphi(Z_1) K_b(Z_1 - z)\}| \\ + \sum_{t=h+1}^T |\text{cov} \{h(Z_t) K_b(Z_t - z) \eta_t, h(Z_1) K_b(Z_1 - z) \eta_1\}|. \quad (\text{A.21})$$

We use the bounds (A.8) and (A.9) in Lemma A.2 to estimate each of the sums in (A.21) separately, which gives

$$T \cdot |A_2| \lesssim h \cdot b^{-d(1-q_G)} + \begin{cases} T^{-q\mathfrak{v}+1} \cdot b^{-d(1+q)} + h^{-\mathfrak{u}+1} \cdot b^{-d}, & q\mathfrak{v} \leq 1 \text{ and } \mathfrak{u} > 1; \\ h^{-q\mathfrak{v}+1} \cdot b^{-d(1+q)} + T^{-\mathfrak{u}+1} \cdot b^{-d}, & q\mathfrak{v} > 1 \text{ and } \mathfrak{u} \leq 1; \\ h^{-q\mathfrak{v}+1} \cdot b^{-d(1+q)} + h^{-\mathfrak{u}+1} \cdot b^{-d}, & q\mathfrak{v} > 1 \text{ and } \mathfrak{u} > 1. \end{cases} \quad (\text{A.22})$$

Thereby using $h \approx b^{-dq_G}$ we obtain

$$T \cdot |A_2| \lesssim b^{-d} + \begin{cases} T^{-q\mathfrak{v}+1} \cdot b^{-d(1+q)} + b^{-d(1+q_G(1-\mathfrak{u}))}, & q\mathfrak{v} \leq 1 \text{ and } \mathfrak{u} > 1; \\ b^{-d(1+q+q_G(1-q\mathfrak{v}))} + T^{-\mathfrak{u}+1} \cdot b^{-d}, & q\mathfrak{v} > 1 \text{ and } \mathfrak{u} \leq 1; \\ b^{-d(1+q+q_G(1-q\mathfrak{v}))} + b^{-d(1+q_G(1-\mathfrak{u}))}, & q\mathfrak{v} > 1 \text{ and } \mathfrak{u} > 1. \end{cases} \quad (\text{A.23})$$

and hence, combining (A.20) and (A.23) gives the bound (A.16) for the term A_2 . \square

We now state a slight variation of Theorem 3.2, where f can satisfy slightly weaker conditions. We use this result to prove Theorem 4.1.

LEMMA A.5. *Suppose the stationary time series $\{Z_t\}$ is 2-mixing with size \mathfrak{v} and Assumption 4.1 is fulfilled for some $q, q_F \in (0, 1)$. Let \hat{f} be defined as in (3.1), where the multiplicative kernel is of order $r > 0$. In addition assume, that the function f belongs to $\mathfrak{G}_{s, \Delta}^d$ for $s, \Delta > 0$ and let $\rho := \min(r, s)$. Then we have*

$$\mathbb{E}|\hat{f}(z) - f(z)| \lesssim b^{2\rho} + b^{-d}T^{-1} + b^{-d(1+q+q_F(1-[(q\mathfrak{v})\vee 1]))}T^{-[(q\mathfrak{v})\wedge 1]}. \quad (\text{A.24})$$

PROOF. Under the stated assumptions using Lemma A.3 the proof is very similar to the proof of Lemma A.4 and we omit the details. \square

PROOF OF THEOREM 4.1. Consider the decomposition

$$\begin{aligned} \hat{\varphi}(z) - \varphi(z) &= \frac{\hat{g}(z)}{\hat{f}(z)} - \frac{\hat{f}(z)}{\hat{f}(z)}\varphi(z) \\ &= \frac{\hat{g}(z) - \hat{f}(z)\varphi(z)}{\hat{f}(z)} + \frac{f(z) - \hat{f}(z)}{\hat{f}(z)} \cdot \frac{\hat{g}(z) - \hat{f}(z)\varphi(z)}{f(z)}. \end{aligned}$$

We first note that Lemma A.5 gives $\mathbb{E}|f(z) - \hat{f}(z)|^2 = o(1)$, which implies that $|\hat{f}(z)^{-1}|$ is bounded in probability. Therefore the second term in the above expansion is of order $o_P(\{\hat{g}(z) - \hat{f}(z)\varphi(z)\}/f(z))$, hence in the decomposition above the second term is negligible in comparison to the first term. Thereby bounding the first term of the decomposition we obtain the result. By using Lemma A.4 and A.5 and noting that $q_{FG} = q_F \wedge q_G$, we obtain Theorem 4.1. \square

PROOF OF COROLLARY 4.2 Under the assumption on the bandwidth the result is obtained by balancing the terms in the bound given in Theorem 4.1. \square

LEMMA A.6. *Suppose the stationary vector time series $\{(X_t, Z_t)\}$ is 2-mixing with size \mathfrak{v} and Assumption 4.2 is fulfilled for some $q, q_G \in (0, 1)$. If $1 \leq t, \tau \leq T$, then*

$$|\text{cov}\{X_t K_b(Z_t - z), X_\tau K_b(Z_\tau - z)\}| \lesssim \min\left(b^{-d(1-q_G)}, b^{-d(1+q)}|t - \tau|^{-q\mathfrak{v}}\right). \quad (\text{A.25})$$

PROOF. Under Assumption 4.2 (ii) the first bound in (A.25) follows from (A.7) in Lemma A.7. On the other hand, under Assumption 4.1 (i,iii) the function $\mathbb{E}[|X_1|^p|Z_1] \cdot f$ is uniformly bounded and $\|K\|_p$ is finite for some $p = 2/(1 - q) > 2$, therefore we have $[\mathbb{E}|X_1 K_b(Z_1 - z)|^p]^{2/p} \lesssim b^{-d(q+1)}$. Using the 2-mixing property of $\{Z_t\}$ together with Hall and Heyde [1980], Theorem A.6, we obtain the second bound in (A.25). \square

LEMMA A.7. Suppose the stationary vector time series $\{(X_t, Z_t)\}$ is 2-mixing with size \mathfrak{v} and Assumption 4.2 is fulfilled for some $q, q_G \in (0, 1)$. Let \hat{g} be defined as in (4.2), where the multivariate kernel is of order $r > 0$. In addition assume, that the function $g = \varphi \cdot f$ belongs to $\mathfrak{G}_{s, \Delta}^d$ for $s, \Delta > 0$ and let $\rho := \min(r, s)$. Then we have

$$\mathbb{E}|\hat{g}(z) - g(z)| \lesssim b^{2\rho} + b^{-d}T^{-1} + b^{-d(1+q+q_G(1-[(q\mathfrak{v}) \vee 1]))}T^{-[(q\mathfrak{v}) \wedge 1]} \quad (\text{A.26})$$

PROOF. Under the stated assumptions using Lemma A.6 the proof is very similar to the proof of Lemma A.4 and we omit the details. \square

PROOF OF THEOREM 4.3. Using Lemma A.5 and A.5 we obtain the result using a similar proof as Theorem 4.1. \square

PROOF OF COROLLARY 4.4 Under the assumption on the bandwidth the result is obtained by balancing the terms in the bound given in Theorem 4.1. \square

A.3 Proofs: Nonparametric regression for panel time series

LEMMA A.8. Suppose $\{X_{t,i}\}$ satisfies (5.1), for all $i, j \in \mathbb{N}$, $\{(X_{t,i}, Z_{t,i}, X_{t,j}, Z_{t,j})\}$ is a stationary time series, and the panel time series $\{(X_{t,i}, Z_{t,i})\}$ is 2-mixing with size \mathfrak{v} and \mathfrak{u} (as defined in Definition 5.1) and Assumption 5.1 is satisfied for some $q_G, q \in (0, 1)$. If $1 \leq t, \tau \leq T$ and $1 \leq i < j \leq N$, then

$$|\text{cov}\{X_{t,i}K_{b_i}(Z_{t,i} - z), X_{\tau,j}K_{b_j}(Z_{\tau,j} - z)\}| \lesssim \min\left((b_i b_j)^{-\frac{d}{2}(1-q_G)}; (b_i b_j)^{-\frac{d}{2}(q+1)}|t - \tau|^{-q\mathfrak{u}}\right), \quad (\text{A.27})$$

while if $1 \leq t, \tau \leq T$ and $1 \leq i \leq N$, then

$$|\text{cov}\{X_{t,i}K_{b_i}(Z_{t,i} - z), X_{\tau,i}K_{b_i}(Z_{\tau,i} - z)\}| \lesssim \min\left(b_i^{-d(1-q_G)}; b_i^{-d(q+1)}|t - \tau|^{-q\mathfrak{u}}\right). \quad (\text{A.28})$$

PROOF. Using Assumption 5.1 together with Hölder's inequality, and recalling that $q_G = 1 - 2/p_G$ with $p_G^{-1} + \bar{p}_G^{-1} = 1$ and $\|G_{t,\tau}^{(i,j)}\|_{p_G}$ is uniformly bounded, we have

$$|\text{cov}\{X_{t,i}K_{b_i}(Z_{t,i} - z), X_{\tau,j}K_{b_j}(Z_{\tau,j} - z)\}| \lesssim (b_i \cdot b_j)^{-d/p_G},$$

where the bound is obtained by using Lyaponov's inequality, which gives $\|K\|_{p_G} < \infty$. This gives the common bound in (A.27) and (A.28). On the other hand, we have $\mathbb{E}[|X_{t,i}K_{b_i}(Z_{t,i} - z)|^p] < \infty$ with $p = 2/(1 - q) > 2$ (Assumption 5.1 (i)). Therefore, using the 2-mixing property of the panel time series $\{(X_{t,i}, Z_{t,i})\}$ together with Hall and Heyde [1980], Theorem A.6, for $i \neq j$, we obtain

$$\begin{aligned} & |\text{cov}\{X_{t,i}K_{b_i}(Z_{t,i} - z), X_{\tau,j}K_{b_j}(Z_{\tau,j} - z)\}| \\ & \lesssim \{\mathbb{E}[|X_{1,i}K_{b_i}(Z_{1,i} - z)|^p] \cdot \mathbb{E}[|X_{1,j}K_{b_j}(Z_{1,j} - z)|^p]\}^{1/p} \cdot |t - \tau|^{-q\mathfrak{u}}, \end{aligned} \quad (\text{A.29})$$

while for $i = j$

$$\begin{aligned} |\text{cov}\{X_{t,i}K_{b_i}(Z_{t,i} - z), X_{\tau,i}K_{b_i}(Z_{\tau,i} - z)\}| \\ \lesssim \{\mathbb{E}[|X_{1,i}K_{b_i}(Z_{1,i} - z)|^p]\}^{2/p} \cdot |t - \tau|^{-q\mathfrak{v}}. \end{aligned} \quad (\text{A.30})$$

Since under Assumption 5.1, the function $g_i^{(p)}(\cdot) = \mathbb{E}[|X_{1,i}|^p | Z_{1,i} = \cdot] f_i(\cdot)$ is uniformly bounded and $\|K\|_p < \infty$ we have

$$\mathbb{E}[|X_{1,i}K_{b_i}(Z_{1,i} - z)|^p]^{1/p} \lesssim b_i^{-\frac{d}{2}(q+1)}. \quad (\text{A.31})$$

Therefore, (A.29) together with (A.31) gives the second bound in (A.27), where (A.30) and (A.31) leads to the second bound in (A.28), which proves the result. \square

LEMMA A.9. Suppose $\{X_{t,i}\}$ satisfies (5.1), for all $i, j \in \mathbb{N}$, $\{(X_{t,i}, Z_{t,i}, X_{t,j}, Z_{t,j})\}$ is a stationary time series, and the panel time series $\{(X_{t,i}, Z_{t,i})\}$ is 2-mixing with size \mathfrak{v} and \mathfrak{u} (as defined in Definition 5.1) and Assumption 5.1 is satisfied for some $q_F, q \in (0, 1)$. If $1 \leq t, \tau \leq T$ and $1 \leq i < j \leq N$, then

$$|\text{cov}\{K_{b_i}(Z_{t,i} - z), K_{b_j}(Z_{\tau,j} - z)\}| \lesssim \min\left((b_i b_j)^{-\frac{d}{2}(1-q_F)}; (b_i b_j)^{-\frac{d}{2}(1+q)} |t - \tau|^{-\mathfrak{u}}\right), \quad (\text{A.32})$$

while if $1 \leq t, \tau \leq T$ and $1 \leq i \leq N$, then

$$|\text{cov}\{K_{b_i}(Z_{t,i} - z), K_{b_i}(Z_{\tau,i} - z)\}| \lesssim \min\left(b_i^{-d(1-q_F)}; b_i^{-d(1+q)} |t - \tau|^{-\mathfrak{v}}\right). \quad (\text{A.33})$$

PROOF. The proof is very similar to the proof of Lemma A.8 and we omit the details. \square

We use the lemma below to prove Theorem 5.1, which requires the following definitions

$$g := \frac{1}{N} \sum_{i=1}^N g_i, \quad \hat{g} := \frac{1}{N} \sum_{i=1}^N \hat{g}_i, \quad f := \frac{1}{N} \sum_{i=1}^N f_i \quad \text{and} \quad \hat{f} := \frac{1}{N} \sum_{i=1}^N \hat{f}_i.$$

LEMMA A.10. Let us suppose that all assumptions in Theorem 5.1 hold. Then we have

$$\begin{aligned} \mathbb{E}|\hat{g}(z) - g(z)|^2 \lesssim \frac{1}{N} \sum_{i=1}^N \left\{ b_i^{2\rho_i} + T^{-1} \cdot b_i^{-d} + T^{-[(q\mathfrak{u}) \wedge 1]} \cdot b_i^{-d(1+q+q_G-q_G[(q\mathfrak{u}) \vee 1])} \right. \\ \left. + N^{-1} \cdot T^{-[(q\mathfrak{v}) \wedge 1]} \cdot b_i^{-d(1+q+q_G-q_G[(q\mathfrak{v}) \vee 1])} \right\} \end{aligned} \quad (\text{A.34})$$

and

$$\begin{aligned} \mathbb{E}|\hat{f}(z) - f(z)|^2 \lesssim \frac{1}{N} \sum_{i=1}^N \left\{ b_i^{2\rho_i} + T^{-1} \cdot b_i^{-d} + T^{-[(q\mathfrak{u}) \wedge 1]} \cdot b_i^{-d(1+q+q_F-q_F[(q\mathfrak{u}) \vee 1])} \right. \\ \left. + N^{-1} \cdot T^{-[(q\mathfrak{v}) \wedge 1]} \cdot b_i^{-d(1+q+q_F-q_F[(q\mathfrak{v}) \vee 1])} \right\}. \end{aligned} \quad (\text{A.35})$$

PROOF. We only give the details for the proof of the MSE of the estimator \hat{g} . The proof of the other result is very similar (but uses the bounds given in Lemma A.9 rather than Lemma A.8) and we omit the details. Consider the standard variance and bias decomposition

$$\mathbb{E}|\hat{g}(z) - g(z)|^2 = \text{var}(\hat{g}(z)) + |\mathbb{E}\hat{g}(z) - g(z)|^2. \quad (\text{A.36})$$

Under the stated assumptions we will derive the following two bounds. The bias is bounded by

$$|\mathbb{E}\hat{g}(z) - g(z)|^2 \lesssim \frac{1}{N} \sum_{i=1}^N b_i^{2\rho_i}, \quad (\text{A.37})$$

while for the variance we have

$$\begin{aligned} \text{var}(\hat{g}(z)) &\lesssim T^{-[(qu)\wedge 1]} \cdot \frac{1}{N} \sum_{i=1}^N [b_i^{-d} + b_i^{-d(1+q+q_G-q_G[(qu)\vee 1])}] \\ &\quad + (NT^{[(qv)\wedge 1]})^{-1} \cdot \frac{1}{N} \sum_{i=1}^N [b_i^{-d} + b_i^{-d(1+q+q_G-q_G[(qv)\vee 1])}]. \end{aligned} \quad (\text{A.38})$$

We now prove (A.37). Using iterative conditional expectation we can write

$$\mathbb{E}\hat{g}(z) = \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T \mathbb{E}(\mathbb{E}[X_{t,i}|Z_{t,i}] K_{b_i}(Z_{t,i} - z)) = \frac{1}{N} \sum_{i=1}^N \int du g_i(u) K_{b_i}(u - z)$$

where $g_i = \mathbb{E}[X_{t,i}|Z_{t,i} = \cdot] f_i(\cdot)$ for all t, i . Since $g = \frac{1}{N} \sum_{i=1}^N g_i$ with $g_i \in \mathfrak{G}_{s_i, \Delta}^d$ and K is a multiplicative kernel of order r with $\int du |u|^r K(u) \leq S_K$, using a Taylor expansion up to order $\rho_i = \min(r, s_i)$ leads to $\mathbb{E}\hat{g}(z) = g(z) + \frac{1}{N} \sum_{i=1}^N b_i^{\rho_i} R_i$ with reminder $|R_i| \leq \Delta S_K < \infty$. Thus applying Jensens inequality we obtain (A.37).

In order to proof (A.38) we consider the expansion

$$\text{var}(\hat{g}(z)) = A_1 + A_2 + A_3 + A_4 \quad (\text{A.39})$$

with

$$\begin{aligned} A_1 &= \frac{1}{N^2 T^2} \sum_{t=1}^T \sum_{i=1}^N \text{var} \{X_{t,i} K_{b_i}(Z_{t,i} - z)\} , \\ A_2 &= \frac{2}{N^2 T^2} \sum_{t=1}^T \sum_{j>i} \text{cov} \{X_{t,i} K_{b_i}(Z_{t,i} - z), X_{t,j} K_{b_j}(Z_{t,j} - z)\} , \\ A_3 &= \frac{4}{N^2 T^2} \sum_{t>\tau} \sum_{j>i} \text{cov} \{X_{t,i} K_{b_i}(Z_{t,i} - z), X_{\tau,j} K_{b_j}(Z_{\tau,j} - z)\} , \\ A_4 &= \frac{2}{N^2 T^2} \sum_{t>\tau} \sum_{i=1}^N \text{cov} \{X_{t,i} K_{b_i}(Z_{t,i} - z), X_{\tau,i} K_{b_i}(Z_{\tau,i} - z)\} . \end{aligned}$$

We will show that $|A_1|, |A_2| \lesssim T^{-1} \cdot \frac{1}{N} \sum_{i=1}^N b_i^{-d}$,

$$|A_3| \lesssim T^{-[(qu) \wedge 1]} \cdot \frac{1}{N} \sum_{i=1}^N [b_i^{-d} + b_i^{-d(1+q+q_G-q_G[(qu) \vee 1])}] \text{ and}$$

$$|A_4| \lesssim (NT^{[(q\mathfrak{v}) \wedge 1]})^{-1} \cdot \frac{1}{N} \sum_{i=1}^N [b_i^{-d} + b_i^{-d(1+q+q_G-q_G[(q\mathfrak{v}) \vee 1])}].$$

Furthermore, if $0 \leq q(\mathfrak{v} \wedge \mathfrak{u}) \leq q/q_G + 1$ then the terms $|A_1|$ and $|A_2|$ are dominated by $|A_3| + |A_4|$. Whereas for $q(\mathfrak{v} \wedge \mathfrak{u}) > q/q_G + 1$ all the terms are of the same order. Therefore, the bound derived for $|A_3| + |A_4|$ will lead to the estimate in (A.38).

First let us consider A_1 . Due to stationarity, we obtain the bound

$$N \cdot T \cdot A_1 \leq \frac{1}{N} \sum_{i=1}^N \mathbb{E}[X_{1,i}^2 K_{b_i}^2(Z_{1,i} - z)].$$

Thereby, under the assumption that the functions $g_i^{(2)}(\cdot) := \mathbb{E}[|X_{1,i}|^2 | Z_{1,i} = \cdot] f_i(\cdot)$ are uniformly bounded and the kernel $\|K\|_2 < \infty$, we have $A_1 \lesssim (N \cdot T)^{-1} \cdot \frac{1}{N} \sum_{i=1}^N b_i^{-d}$.

It is straightforward to show that $T \cdot |A_2|$ is bounded by

$$\frac{2}{N^2} \sum_{j>i} \text{var}(X_{1,i} K_{b_i}(Z_{1,i} - z))^{1/2} \text{var}(X_{1,j} K_{b_j}(Z_{1,j} - z))^{1/2} \lesssim \frac{1}{N^2} \sum_{j>i} (b_i b_j)^{-d/2},$$

where the inequality above follows by applying the same arguments as those used for A_1 . Therefore using the Cauchy-Schwarz inequality we obtain $|A_2| \lesssim T^{-1} \cdot \frac{1}{N} \sum_{i=1}^N b_i^{-d}$.

The term $T \cdot |A_3|$ is bounded by the sum

$$\frac{8}{N^2} \sum_{j>i} \sum_{t=2}^T |\text{cov}\{X_{t,i} K_{b_i}(Z_{t,i} - z), X_{1,j} K_{b_j}(Z_{1,j} - z)\}|.$$

We now derive bounds for $T \cdot |A_3|$ for different mixing sizes. If $qu \leq 1$ then we estimate the inner sum using the second bound in (A.27) of Lemma A.8, i.e., $T \cdot |A_3| \lesssim \frac{1}{N^2} \sum_{j>i} (b_i b_j)^{-\frac{d}{2}(q+1)} T^{-qu+1}$, which leads together with the Cauchy-Schwarz inequality to $|A_3| \lesssim T^{-qu} \frac{1}{N} \sum_{i=1}^N b_i^{-d(q+1)}$. On the other hand, if $qu > 1$ then we partition the inner sum into two parts which we estimate separately using now the two bounds in (A.27) of Lemma A.8, thus giving us

$$T \cdot |A_3| \lesssim \frac{1}{N^2} \sum_{j>i} \left\{ \sum_{t=2}^{h_{ij}} (b_i b_j)^{-\frac{d}{2}(1-q_G)} + \sum_{t=h_{ij}+1}^T (b_i b_j)^{-\frac{d}{2}(q+1)} t^{-qu} \right\}.$$

Thereby using $h_{ij} \approx (b_i b_j)^{-\frac{d}{2}q_G}$ together with the Cauchy-Schwarz inequality we obtain $T \cdot |A_3| \lesssim \frac{1}{N} \sum_{i=1}^N [b_i^{-d} + b_i^{-d(1+q+q_G(1-qu))}]$. Combining the bounds in the two cases $qu \leq 1$ and $qu > 1$ we have $|A_3| \lesssim T^{-[(qu) \wedge 1]} \cdot \frac{1}{N} \sum_{i=1}^N [b_i^{-d} + b_i^{-d(1+q+q_G-q_G[(qu) \vee 1])}]$.

The term $N \cdot T \cdot |A_4|$ is bounded by

$$\frac{4}{N} \sum_{i=1}^N \sum_{t=2}^T |\text{cov} \{X_{t,i} K_{b_i}(Z_{t,i} - z), X_{1,i} K_{b_i}(Z_{1,i} - z)\}|.$$

We estimate the inner sum applying the same arguments as those used for $|A_3|$ (but use the bounds given in (A.28) rather than (A.27) of Lemma A.8). Thereby we obtain $N \cdot |A_4| \lesssim T^{-[(q\mathbf{v}) \wedge 1]} \cdot \frac{1}{N} \sum_{i=1}^N [b_i^{-d} + b_i^{-d(1+q+q_G-q_G[(q\mathbf{v}) \vee 1])}]$. \square

PROOF OF THEOREM 5.1. The proof is very similar to the proof of Theorem 4.1 (but uses Lemma A.10 rather than Lemma A.4 and A.5) and we omit the details. \square

PROOF OF COROLLARY 5.2. Under the assumptions of the corollary we have

$$|\hat{\varphi}(z) - \varphi(z)|^2 = O_p \left(\frac{1}{N} \sum_{i=1}^N \left\{ b_i^{2\rho_i} + T^{-1} \cdot b^{-d} + T^{-[(q\mathbf{u}) \wedge 1]} \cdot b_i^{-d(1+q+q_{FG}-q_{FG}[(q\mathbf{u}) \vee 1])} \right\} \right), \quad (\text{A.40})$$

applying Theorem 5.1, where $\frac{1}{N} \sum_{i=1}^N N^{-1} \cdot T^{-[(q\mathbf{u}) \wedge 1]} \cdot b_i^{-d(1+q+q_{FG}-q_{FG}[(q\mathbf{u}) \vee 1])}$ is asymptotically negligible when $\mathbf{v} \geq \mathbf{u}$. Under the assumption on the bandwidths the result is obtained balancing the three terms in the bound (A.40). \square

PROOF OF COROLLARY 5.3. The result is obtained applying Theorem 5.1, where given $\mathbf{v} \leq \mathbf{u}$ the assumption on the bandwidths provides the balance of the four terms of the bound (5.5). \square

PROOF OF COROLLARY 5.4. Under the assumptions of the corollary we apply Corollary 5.3, where given $N^{\zeta_i} \approx T^{(\gamma_i - \delta_i)}$ for some $\zeta_i > 0$ the bound (5.8) simplifies to

$$|\hat{\varphi}(z) - \varphi(z)|^2 = O_P \left(\frac{1}{N} \sum_{i=1}^N \left[T^{\left\{ \frac{\gamma_i - \delta_i}{\gamma_i} - \frac{\gamma_i - \delta_i}{\delta_i} \right\} \cdot [(q\mathbf{u}) \wedge 1]} + T^{-\frac{\gamma_i - \delta_i}{\zeta_i} + \frac{\gamma_i - \delta_i}{\delta_i} \cdot [(q\mathbf{v}) \wedge 1]} \right] \cdot T^{-\frac{2\rho_i}{2\rho_i + d} \cdot \gamma_i} \right).$$

Since $\mathbf{v} \leq \mathbf{u}$ for each $i \in \mathbb{N}$ implies $T^{\left\{ \frac{\gamma_i - \delta_i}{\gamma_i} - \frac{\gamma_i - \delta_i}{\delta_i} \right\} \cdot [(q\mathbf{u}) \wedge 1]} = O(1)$. We obtain the result, if for each $i \in \mathbb{N}$ we have $T^{-\frac{\gamma_i - \delta_i}{\zeta_i} + \frac{\gamma_i - \delta_i}{\delta_i} \cdot [(q\mathbf{v}) \wedge 1]} = O(1)$ or equivalently $\delta_i / [(q\mathbf{v}) \wedge 1] \geq \zeta_i$, which is just the condition given in Corollary 5.4. \square

A.4 Covariances and 2-mixing rates for linear processes

We use the results derived in this section in Section 3.1, where we compared the rates of convergence for linear processes with the rates in the general 2-mixing case.

Let us suppose $\{Z_t\}$ satisfies the linear process representation in (3.6). By placing some additional conditions on the innovations we have the following lemma, which is due to Giraitis et al. [1996], Lemma 1 and 2.

LEMMA A.11 (Giraitis et al. [1996]). Suppose $\{Z_t\}$ is a linear process which satisfies (3.6), and $\text{cov}(Z_0, Z_t) \leq Ct^{-\theta}$. Let f be the density of Z_t and f_t denote the joint density Z_0, Z_t . If $\mathbb{E}(|\varepsilon_t^3|) < \infty$, and for all $u \in \mathbb{R}$ suppose the characteristic function satisfies $|\mathbb{E}[\exp(-iu\varepsilon_1)]| \leq \frac{1}{(1+|u|)^\delta}$ for some $\delta > 0$, then the joint density satisfies the relation

$$f_t(x, y) = f(x)f(y) + r(t)f'(x)f'(y) + O(t^{-\theta-d}),$$

where $f' \in L_1(\mathbb{R})$ and $r(t) = \text{cov}(Z_0, Z_t)$, for some $0 < d < \min(\frac{\theta}{7}, \frac{1-\theta}{12})$.

Using the result above the MSE of the kernel estimator with observations from a linear process can be derived.

For most processes, there isn't a direct correspondence between the 2-mixing and the covariance size. However for Gaussian processes both sizes are linked by the inequality

$$\frac{|\text{cov}(X_0, X_t)|}{\text{var}(X_0)} \leq \sup_{A \in \sigma(Z_0), B \in \sigma(Z_t)} |P(A \cap B) - P(A)P(B)| \leq 2\pi \frac{|\text{cov}(X_0, X_t)|}{\text{var}(X_0)} \quad (\text{A.41})$$

(see Doukhan [1994], Section 2.1), thus the covariance and the 2-mixing sizes are the same. Suppose that $\{Z_t\}$ satisfies (3.6), where the innovations are Gaussian and $|a_j| \lesssim j^{-\theta}$. Then we have

$$\left. \begin{array}{c} \frac{|\text{cov}(X_0, X_t)|}{\text{var}(X_0)} \\ \text{and} \\ \sup_{A \in \sigma(Z_0), B \in \sigma(Z_t)} |P(A \cap B) - P(A)P(B)| \end{array} \right\} \lesssim \begin{cases} (\log t)t^{-(2\theta-1)}, & \text{if } 1/2 < \theta \leq 1; \\ t^{-\theta}, & \text{if } \theta > 1. \end{cases} \quad (\text{A.42})$$

We now consider more general linear processes, which are not necessarily Gaussian. Then the covariance size does not immediately give the 2-mixing size. However, if the density of the innovations satisfies certain smoothness conditions then we can obtain the following bound.

LEMMA A.12. Suppose $\{Z_t\}$ is a linear process which satisfies the representation $Z_t = \sum_{j=0}^{\infty} a_j \varepsilon_{t-j}$, where the parameters $|a_j| \leq Cj^{-\theta}$ and $\theta > 1/2$. Let f_ε be the density of the innovation ε_t . If $\mathbb{E}(|\varepsilon_t|^\ell) < \infty$ (where $\ell > 2$) and $\int |f_\varepsilon(x+a) - f_\varepsilon(x)|dx \leq C|a|$, then we have

$$\sup_{A \in \sigma(Z_0), B \in \sigma(Z_t)} |P(A \cap B) - P(A)P(B)| \leq Cj^{(-2\theta+1)\frac{\ell}{2(\ell+1)}}$$

where C are some arbitrary constants.

PROOF. The result can be proved using a straightforward adaptation of the proofs in Chanda [1974], Gorodetskii [1977] and Davidson [1994] (Theorem 14.9), who proved the result for strong α -mixing. Hence we omit the details. \square

REMARK A.1. It is interesting to compare the 2-mixing sizes derived in Lemma A.12 with the strong α -mixing results for MA(∞) processes. Under the same set of conditions, but with the additional restriction that $\theta > 3/2$, we have that

$$\sup_{A \in \sigma(Z_0, Z_{-1}, \dots), B \in \sigma(Z_t, Z_{t+1}, \dots)} |P(A \cap B) - P(A)P(B)| \lesssim |t|^{(-2\theta+1)\frac{\ell}{2(\ell+1)+1}}.$$

In other words, the 2-mixing size is larger than the α -mixing size. This is because, by definition, the σ -algebras involved in the definition of α -mixing is far larger than the σ -algebras in the definition of 2-mixing, thus allowing more extreme cases. \square

Comparing Lemma A.12 with the covariance size given in (A.42) we see when the Gaussianity assumption is relaxed the covariance and 2-mixing sizes no longer coincide. However by using Lemma A.12 and Hall and Heyde [1980], Theorem A.5, we have the upper and lower bounds

$$j^{(-2\theta+1)\frac{\ell}{(\ell-2)}} \lesssim \sup_{A \in \sigma(Z_0), B \in \sigma(Z_t)} |P(A \cap B) - P(A)P(B)| \lesssim j^{(-2\theta+1)\frac{\ell}{2(\ell+1)}}.$$

Therefore the 2-mixing size \mathfrak{v} of the linear process $\{Z_t\}$ is bounded by

$$(2\theta - 1)\frac{\ell}{2(\ell + 1)} \leq \mathfrak{v} \leq (2\theta - 1)\frac{\ell}{(\ell - 2)}. \quad (\text{A.43})$$

References

- K. B. Athreya and S. Pantula. Mixing properties of Harris chains and autoregressive processes. *J. Appl. Probab.*, 23:880–892, 1986.
- B. Baltagi. *Econometric Analysis of Panel Data*. John Wiley and Sons, Chicester, 2nd edition, 2001.
- B. Basrak, R. Davis, and T. Mikosch. Regular variation of GARCH processes. *Stochastic Processes and their Applications*, 99:95–115, 2002.
- D. Bosq. *Nonparametric Statistics for Stochastic Processes*. Springer, New York, 1998.
- F. Bousamma. *Ergodicité, mélange et estimation dans les modèles GARCH*. PhD thesis, Paris 7, 1998.
- R. C. Bradley. A covariance inequality under a two-part dependence assumption. *Statistics and Probability Letters*, 30:287–293, 1996.
- K. C. Chanda. Strong mixing properties of linear stochastic processes. *J. Appl. Prob.*, 11: 401–408, 1974.
- B. Cheng and P. M. Robinson. Density estimation in strongly dependent non-linear time series. *Statistica Sinica*, 1:335–359, 1991.
- B. Cheng and P. M. Robinson. Semiparametric estimation from time series with long-range dependence. *Journal of Econometrics*, 94:335–353, 1994.
- D. Cline and H. Pu. Geometric ergodicity of nonlinear time series. *Statistica Sinica*, 9: 1103–118, 1999.

- S. Csörgö and J. Mielniczuk. Nonparametric regression under long-range dependent normal errors. *Ann. Statist.*, 23:1000–1014, 1995.
- S. Csörgö and J. Mielniczuk. The smoothing dichotomy in random-design regression with long-memory based on moving averages. *Statistica Sinica*, 10:771–787, 2001.
- S. Csörgö and J. Mielniczuk. Random-design regression under long range dependence errors. *Bernoulli*, 5:209–224, 1999.
- R. Dahlhaus and S. Feiler. Panel time series. *Preprint*, 2005.
- J. Davidson. *Stochastic Limit Theory*. Oxford University Press, Oxford, 1994.
- P. Doukhan. *Mixing, Properties and Examples*. Springer, New York, 1994.
- G. Estevás and P. Vieu. Nonparametric estimation under long memory dependence. *Nonparametric Statistics*, 15:535–551, 2003.
- J. Geweke and S. Porter-Hudak. The estimation and application of long memory time series models. *Journal of Time Series Analysis*, 4:221–238, 1983.
- L. Giraitis, H. L. Koul, and D. Surgailis. Asymptotic normality of regression estimators with long memory errors. *Statistics and Probability Letters*, 29:317–335, 1996.
- L. Giraitis, P. Kokoszka, and R. Leipus. Stationary ARCH models: Dependence structure and central limit theorem. *Econometric Theory*, 16:3–22, 2000.
- V. Gorodetskii. On the strong mixing property for linear sequences. *Theory of Probability and its Applications*, 22:411–413, 1977.
- P. Hall and J. Hart. Convergence rates in density estimation for data from infinite-order moving average processes. *Probability Theory and Related Fields*, 87:253–274, 1990a.
- P. Hall and J. Hart. Nonparametric regression with long-range dependence. *Stochastic Processes and their Applications*, 87:339–351, 1990b.
- P. Hall and C. Heyde. *Martingale Limit Theory and its Application*. Academic Press, New York, 1980.
- V. Hjellvik and D. Tjøstheim. Modelling panels of intercorrelated autoregressive time series models. *Biometrika*, 86:573–590, 1999.
- V. Hjellvik, R. Chen, and D. Tjøstheim. Nonparametric estimation and testing in panels of intercorrelated time series models. *Journal of Time Series Analysis*, 25:831–872, 2004.
- J. Johannes, M. Jun, and S. Subba Rao. Nonparametric estimation of spatio-temporal covariance functions. *Preprint*, 2007.

- H. R. Künsch. Statistical aspects of self-similar processes. *Proceedings of the World Congress of the Bernoulli Society*, 1:67–74, 1987.
- O. B. Linton and E. Mammen. Estimating semiparametric ARCH(∞) models by kernel smoothing methods. *Econometrica*, 73:771–836, 2004.
- E. Mammen, B. Støve, and D. Tjøstheim. Nonparametric additive models for panels of time series. *Preprint*, 2005.
- E. Masry and D. Tjøstheim. Nonparametric estimation and identification of nonlinear ARCH time series. *Econometric Theory*, 11:258–289, 1995.
- J. Mielniczuk. On the asymptotic mean integrated squares error of kernel density estimator for dependent data. *Statistics and Probability Letters*, 34:53–58, 1997.
- E. Rio. Covariance inequalities for strongly mixing processes. *Ann. Inst. H. Poincaré Prob. Statist.*, 29:587–597, 1993.
- P. M. Robinson. Testing for strong serial correlation and dynamic conditional heteroskedasticity in multiple regression. *Journal of Econometrics*, 47:67–78, 1991.
- P. M. Robinson. Log-periodogram regression of time series with long range dependence. *Ann. Statist.*, 23:1048–1072, 1995.
- M. Rosenblatt. Density estimates and markov sequences. In M. Puri, editor, *Non-parametric techniques in statistical inference*, pages 199–210. Cambridge University Press, London, 1970.
- D. W. Scott. *Multivariate Density Estimation*. Wiley, New York, 1992.
- S. Subba Rao. A note on uniform convergence of an ARCH(∞) estimator. *Sankhya*, 68:600–620, 2006.
- S. Subba Rao. On mixing properties of ARCH and time-varying ARCH processes. *Preprint*, 2007.